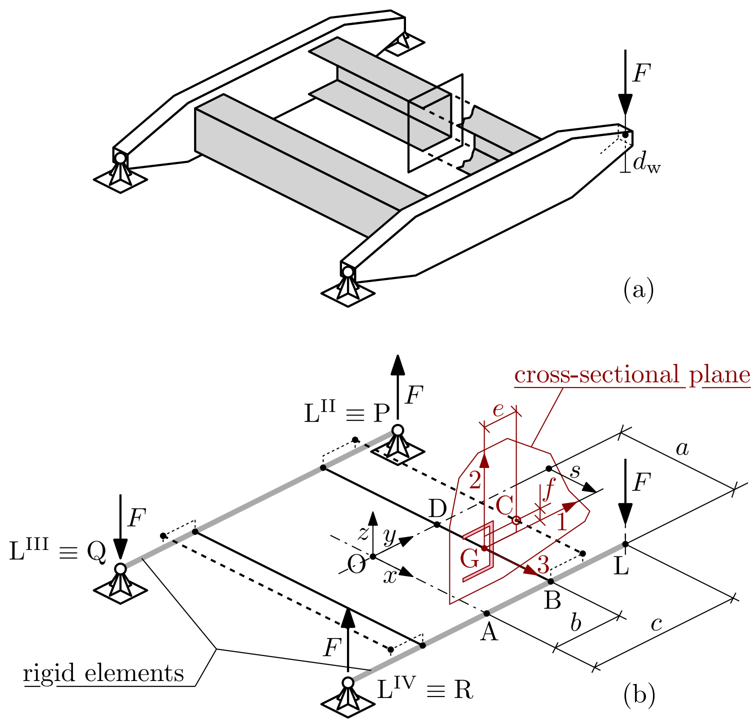

Figure 1: A simplified ladder-frame chassis, consisting in two longitudinal channel-section beams spanning along the wheelbase; their connection to the axles are assumed as rigid for simplicity, and the three supports are such to exert a purely vertical reaction force.

0.1 A semi-worked example: a simplified ladder frame chassis

The present contribution concerns the *torsional stiffness*[1] evaluation for the simplified ladder-frame chassis depicted in Fig. 1, whose track width is $2c$ for both the axles, and whose wheelbase is $2a$; the length of the two rail profiles is nominally assumed equal to the wheelbase.

Torsional stiffness is an established chassis structure conventional property, which is significant for the suspension tuning practicability with respect to under-/oversteering control; still, it is simplistic to

[1]a.k.a. torsional *rigidity*

assume that a high enough torsional stiffness may prevent handling issues in general, since, e.g. it is pretty uncoupled with the structure response to dynamic lateral forces. Nevertheless, the measurement procedure is straightforward, and the test rig is cheap.

Fig. 1a represents a formally correct test setting for the torsional stiffness; the chassis is *simply* supported at three of the wheel centers, whereas a vertical force F is applied at the fourth one, at which the d_{w} vertical deflection is also measured.

The vertical supports allow for three residual rigid body motions along the (O,xy) horizontal plane; a statically determinate set of further constraint is required for uniquely positioning the structure in space.

It is of the most importance to grant the statically determinate nature of the overall constraining system, since any further restraint might unduly support the loaded structure, and thus spuriously raising its observed stiffness.

A straightforward analysis of the chassis structure global equilibrium[2] returns that each axle is loaded by a pair of equal and opposite vertical forces - i.e. by a pure, longitudinally oriented moment vector, and that those two front and rear moments are self compensating. In the case of equal track widths, four vertical forces of equal magnitude F are applied at the four wheel centers, whose orientation switches along the axles, and from the axle to axle; in the case of different track widths, forces of equal magnitude are applied at each wheel of the axle, and they scale from the front to the rear with the inverse of the track width.

Once obtained the experimental ratio between the F force and the d_{w} deflection, the torsional stiffness k may be derived as the ratio between the magnitude of the torque applied to each axle, and the relative twist angle, namely

$$k = \frac{2cF}{\frac{d_{\mathrm{w}}}{2c}} = \frac{F(2c)^2}{d_{\mathrm{w}}}, \tag{1}$$

[2]with reference to Fig. 1b, i) rotational equilibrium with respect to the rear PQ axle requires a downward F force at R, ii) rotational equilibrium with respect to the front LR axle requires that the two rear supports exert equal and opposite vertical reactions, whose magnitude is set by iii) the rotational equilibrium with respect to the longitudinal chassis axis.

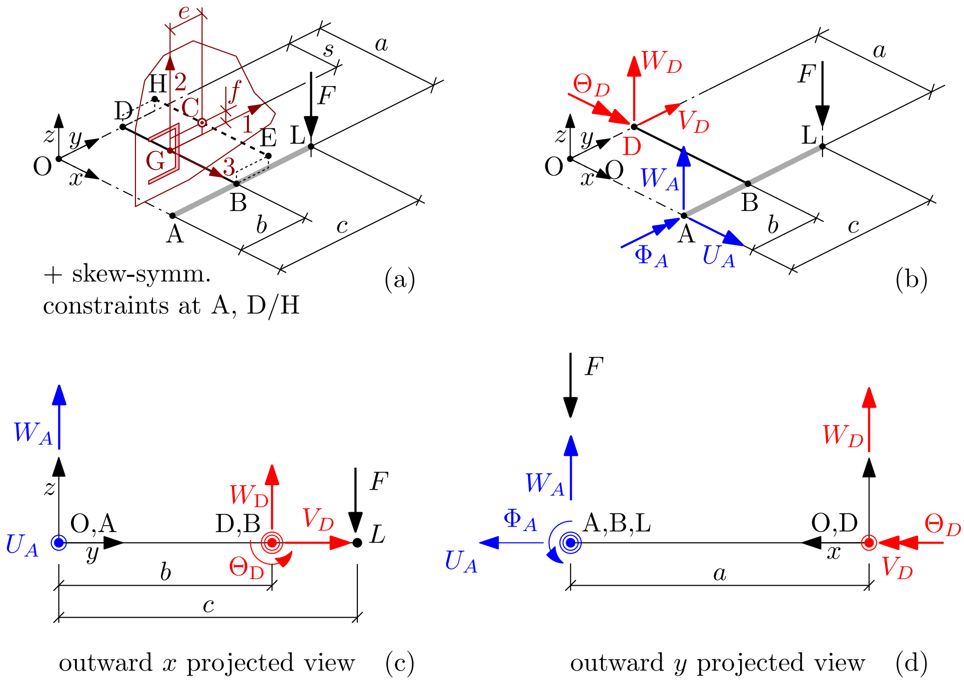

Figure 2: A quarter portion of the ladder frame, which is the minimal portion to be modeled due to the dual skew-symmetry. Please note that the (O,xyz) reference system of the present figure coincides with its fixed counterpart as in Fig. 1 in the undeformed configuration only. The present figure reference frame partly follows, in fact, the structure deflection.

where the $2c$ track width pertains to the axle that includes the loaded (and monitored in deflection) wheel center.

In the case under scrutiny of equal track widths, the twice-symmetric structure is loaded by a system of four forces which are skew-symmetrically arranged with respect to both the (O,zx) and the (O,yz) planes[3].

A twice skew-symmetric problem is thus obtained, whose representative portion - a quarter of the whole structure - is represented Figure 2a.

[3]a third skew-symmetry plane, namely the (O,xz) exists if the profiles are consistently symmetric; however, limited benefit is attained in considering such a third skew-symmetry plane in the treatise.

Skew-symmetry constraints are required at the intersection of the front axle rigid member with the (O,zx) plane - a point, this, which is *nominally* embodied by the A location[4], and at the intersection of the longitudinal rails with the (O,yz) plane, nominally taken at the D centroid of the interested cross section. Those constraints are set in order to grant material - or rigid body motion law - continuity between the modeled, representative portion of the structure, and its images, and they lead to the reaction forces and moments listed in Fig. 2b.

It worth to mention that the problem depicted in Fig. 1 is *not* twice skew-symmetric in itself, due to the unsimmetric nature of the support arrangement; however, the problem acquires such a property once the exerted reaction forces are considered, in place of the originating constraints.

In such cases, the problem solutions obtained i) for the complete structure, subject to the original constraints, and ii) for the representative portion, and duly mirrored, are consistent in terms of strains, stresses and with regard to their resultants, whereas they differ by a rigid body motion in terms of absolute displacement and rotations. Such a behavior can be rationalised by considering that a moving frame exists, according to which a [skew-]symmetric structure behavior is observed; this moving reference system is ideally pinned to the structure at the same d.o.f.s that are affected by the [skew-]symmetry constraints.

Most of the skew-symm. constraint reaction forces may be set based on the equilibrium equations for the quarter ladder-frame structure, see Fig. 2b; U_{A} and V_{D} are set null based on the translational equilibrium with respect to the global x and y axes, respectively. By casting a system of equations which involves the translational equilibrium with respect to z, and the rotational equilibrium with respect to the (O,x) and the (O,y) axes - see Figs 2c and 2d, other three unknown reactions amongst W_{A}, Φ_{A}, W_{D} and Θ_{D} may be defined; the remaining independent equilibrium equation - a rotational one, and associated to the (O,z) axis - is trivially satisfied in the absence of any contribution, thus making the overall system of equations rank deficient of degree one.

The [quarter] ladder-frame structure, loaded according to the torsional stiffness test, appears hence once *internally* statically indetermi-

[4]any point of the (O,zx) plane may equally serve the purpose

nate[5]; we then define a principal structure by fictively releasing the z oriented constraint at A, and thus allowing for a z-oriented slippage at the interface between the ABL rigid member and its image. As usual, the associated W_{A} reaction force is treated as a parameter, whose value is tuned to reinstate continuity at A; the remaining constraint reactions are then obtained as a linear combination of the F and W_{A} load parameters.

The second Castigliano theorem is employed to evaluate the w_{A} vertical deflection at A, which in turn requires the expression for the internal strain energy to be cast as a function of the same aforementioned load parameters.

Since the contribution of a rigid member to the structure strain energy is zero by definition, we proceed to the evaluation of the internal action components for the channel section rail; by considering the equilibrium of a DG rail segment, where G is a centroidal point taken at a s distance from D - see Fig. 2a, we obtain

$$N = 0 \qquad Q_1 = -V_{\mathrm{D}} = 0 \qquad Q_2 = -W_{\mathrm{D}} = W_{\mathrm{A}} - F$$

and

$$\begin{aligned} M_1 &= -sW_{\mathrm{D}} = -s\left(W_{\mathrm{A}} - F\right) \\ M_2 &= +sV_{\mathrm{D}} = 0 \\ M_{\mathrm{t}} &= -\Theta_{\mathrm{D}} + eW_{\mathrm{D}} - fV_{\mathrm{D}} = \left(F - W_{\mathrm{A}}\right)(e+b) - Fc. \end{aligned}$$

The torsional moment M_{t} does not coincide with $-\Theta_{\mathrm{D}}$ since the $V_{\mathrm{D}}, W_{\mathrm{D}}$ shear aligned forces are assumed as applied at the D - which is a centroid, and they result shifted with respect to the cross-sectional shear center.

The following expression for the strain energy lineic density is employed – cfr. Eq.**??** – which preserves the contribution of all the internal action components

$$\begin{aligned} \frac{dU}{dl} = & \frac{N^2}{2EA\alpha_{\mathrm{axl}}} + \frac{J_{22}M_1^2 + J_{11}M_2^2 + 2J_{12}M_1M_2}{2E\left(J_{11}J_{22} - J_{12}^2\right)\alpha_{\mathrm{flx}}} \\ & + \frac{M_{\mathrm{t}}^2}{2GK_t\alpha_{\mathrm{trs}}} + \frac{\chi_1Q_1^2 + \chi_2Q_2^2 + \chi_{12}Q_1Q_2}{2GA\alpha_{\mathrm{shr}}}. \end{aligned}$$

[5]Please note that, apart from the peculiar [skew-]symmetric condition, a spatial closed ring is in general six times statically indeterminate.

Also, the cross section elastic characteristic with respect to each internal action component is scaled by a normally unit auxiliary factor $\alpha_\Box$, which may steer the elastic response towards infinite compliance ($\alpha_\Box \to 0$) or infinite stiffness ($\alpha_\Box \to \infty$); those stiffness multipliers will be employed in the discussion of the results below, and may be ignored otherwise.

We may now integrate such a lineic strain energy density over the interval $s \in [0, a]$, thus obtaining the U internal strain energy for the quarter frame as a quadratic function of F and W_A.

The vertical deflection at A may then be derived according to the Castigliano theorem, and may be set to zero in order to obtain the expression of W_A as a linear function of F.

As in the previous worked example, such $W_\mathrm{A}(F)$ is substituted within the $U\,(F, W_\mathrm{A})$ quarter frame internal energy expression, which becomes a function of the sole external load F.

A further applications of the Castigliano theorem let us derive the d deflection of the F force application point for the quarter ladder-frame structure, i.e. with respect to the aforementioned moving reference system, according to which the displacement field is twice skew-symmetric.

The absolute d_w deflection of the L wheel center, i.e. the deflection observed according to a reference system consistent with the three supports of Fig. 1, may be derived based on the observation that the internal energy for the *whole* chassis is four times the one evaluated for the quarter structure; we thus obtain

$$d_\mathrm{w} = \frac{d(4U)}{dF} = 4d. \tag{2}$$

Such a result may be rationalized considering the $(-d, +d, -d, +d)$ vertical deflections of the (L,P,Q,R) points, respectively, derived from the mirrored quarter structure response. A downward, uniform, translation of magnitude d reestablishes the congruence with the fixed supports at points P,R, with overall deflections $(-2d, 0, -2d, 0)$. Finally, a suitable rotation with respect to the PR diagonal raises of a $2d$ quantity the vertical position of the Q point, thus reinstating compliance with the third support. Since the L and the Q points are located at an equal distance from the pivot line, the same rotation lowers the L point of an equal $2d$ quantity, thus leading to the absolute deflection configuration $(-4d, 0, 0, 0)$.

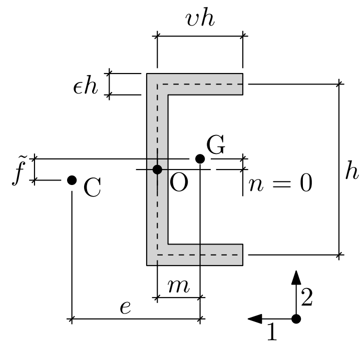

Figure 3: Dimensions for the channel section employed in calculations. The web height h is the relevant dimensional parameter, whereas υ is the ratio between the flange width and the web height, and ϵ sets the ratio between the wall thickness – assumed uniform – and the web height. Centroidal coordinates (m, n) are measured with respect to the web midspan, whereas the shear center coordinates (e, f) are measured with respect to the centroid. Please note that the cross-section properties reported in the maxima worksheet are evaluated according to a small thickness hypothesis, i.e. their expressions is consistent with a vanishing ϵ, first order Taylor expansion.

The ladder-frame chassis torsional stiffness may be then evaluated according to Eqn. 1, which leads to a pretty composite expression whose rationalization is complicated.

In order to isolate the influence of the various parameters, a reference configuration is defined in terms of channel section and global chassis dimensions. In particular, the worked example provided in form of a maxima worksheet employs the channel section of Fig.3 for both the rails. Also, all the $\alpha_\square$ auxiliary stiffness multipliers are bonded to unity in the reference case.

The response of the structure to the variation of one or more parameters is assessed based on the torsional stiffness ratio between the altered configuration, and the reference one.

Leaving to the willing reader the influence analysis of the various parameters[6], we focus on how the three main sources of compliance –

[6]Consider in particular the influence of the rail span length a, of their mutual distance b, and the influence of the cross section size h and thickness t. Since the

namely, the rail compliance to the torsional moment, to the bending moment, and to shear actions – interact in defining the overall compliance of the simple structure under scrutiny.

The ratio is hence considered between the chassis torsional stiffness for the reference configuration, namely k_{ref}, and its $k\left(\alpha_{\text{trs}}, \alpha_{\text{flx}}, \alpha_{\text{shr}}\right)$ counterpart obtained with given α_{trs}, α_{flx} and α_{shr} profile torsional, bending a shear stiffness multipliers, respectively.

If we speculate the profile flexural[7] and shear stiffness to vanish, i.e. the longitudinal rail torsional stiffness is fictitiously left alone in elastically connecting the two rigid elements, we may consider the limit

$$\lim_{\alpha_{\text{flx}}, \alpha_{\text{shr}} \to 0} \frac{k\left(1, \alpha_{\text{flx}}, \alpha_{\text{shr}}\right)}{k_{\text{ref}}} = p > 0 \tag{3}$$

which returns a nonzero fraction of the unity, whose value is pretty small for the open thin-walled section under scrutiny, but that might become relevant for bulky closed section profiles. Each profile is in fact twisted by the same amount of relative rotation that occurs between the two front and rear rigid members, thus accumulating internal strain energy, and thus requiring a finite work-supplying external force.

We now consider the complementary condition, in which the two rails lose their capability to elastically react to torsion, whilst retaining their full shear and flexural stiffness; we hence consider the limit

$$\lim_{\alpha_{\text{trs}} \to 0} \frac{k\left(\alpha_{\text{trs}}, 1, 1\right)}{k_{\text{ref}}} = 1 - p > 0 \tag{4}$$

which also returns a nonzero fraction of the unity, complementary to the former one, as expected. Such a fraction of the overall stiffness may be observed to vanish e.g. with vanishing spacing between the two rails; it is in fact associated to the vertical misalignment of the longitudinal beam ends, which equates the product of the relative rigid member rotation by an arm that is equal to half the rail spacing. Such

material is homogeneous and isotropic, the stiffness varies linearly with the Young modulus, you don't really have to check. For a cleaner analysis, try also to isolate the various sources of compliance, i.e. compliance with respect to bending moments, torsional moment, and shear actions alone.

[7]Here, we follow for added clarity the academic distinction between *bending*, which – in consistency with to the general *nonuniform* bending meaning – may be employed as an umbrella term for both flexure and shear, and *flexure*, which excludes shear contributions.

a vertical misalignment may not be achieved through a profile rigid motion, and hence a further contribution is due to the overall strain energy.

The complementary nature of the two above quantities hints for a in parallel disposition of the two means the profile may react to the relative rotation of the rigid members they are clamped to.

The latter contribution may be further scrutinized by splitting the two distinct shear and flexural contributions; we consider in particular the two limits

$$\lim_{\alpha_{\mathrm{shr}}\to 0}\lim_{\alpha_{\mathrm{trs}}\to 0}\frac{k\left(\alpha_{\mathrm{trs}},1,\alpha_{\mathrm{shr}}\right)}{k_{\mathrm{ref}}}=0 \quad \lim_{\alpha_{\mathrm{flx}}\to 0}\lim_{\alpha_{\mathrm{trs}}\to 0}\frac{k\left(\alpha_{\mathrm{trs}},\alpha_{\mathrm{flx}},1\right)}{k_{\mathrm{ref}}}=0 \quad (5)$$

which both vanish, thus indicating that none of the two isolated elastic reactions may be activated alone. By turning into rigid the profile with respect to either shear or flexure, i.e.

$$\lim_{\alpha_{\mathrm{shr}}\to \infty}\lim_{\alpha_{\mathrm{trs}}\to 0}\frac{k\left(\alpha_{\mathrm{trs}},1,\alpha_{\mathrm{shr}}\right)}{k_{\mathrm{ref}}}=q>0 \quad (6)$$

$$\lim_{\alpha_{\mathrm{flx}}\to \infty}\lim_{\alpha_{\mathrm{trs}}\to 0}\frac{k\left(\alpha_{\mathrm{trs}},\alpha_{\mathrm{flx}},1\right)}{k_{\mathrm{ref}}}=r>0, \quad (7)$$

we obtain finite normalized flexural and shear compliances, $1/q$ and $1/r$ respectively, whose sum equates the cumulative compliance $1/(1-p)$ attributable to the overall bending. An in-series arrangement of the two flexural and shear compliances is thus suggested, which finds rationalization in the fact that the same end transverse shift may be accomodated both through i) an S-shaped, purely flexural deflection, ii) through a card-deck pure shear inclination, or iii) a combination of the two.

The compliance components' arrangement for the simplified ladder-frame chassis under scrutiny is shown in Figure 4; in a general structure, the mutual interaction of the elastic members may not be pigeonholed within the simplistic "many in parallel" or "many in-series" models; a complex combination of those two elementary modes may be considered even for the simple, single d.o.f. treated in the present paragraph.

It is finally noted that, in the case of struts manufactured from composite laminates, the speculative selective deactivation of one or the other mean of elastic response earns actual significance, since the simple lack of dedicated laminae may suffice in obtaining such a tricky behavior.

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\\|      trs.compliance      +----+
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Figure 4: An ASCII art rationalization of the compliance components' arrangement for the simplified ladder frame chassis under scrutiny; the torsional elastic compliance of the profiles acts in parallel with their combined in-series flexural and shear compliance.