0.1 Basic formulation for plates and shells

A necessary condition for applying the plate/shell model framework to a deformable body is that a geometrical midsurface might be, if only loosely, recognized for such a body. Then, an iterative refinement procedure[1] may be applied to such tentative midsurface guess.

Then, material should be observed as *[piecewise-]*homogeneous, or slowly varying in mechanical properties while moving at a fixed distance from the midsurface.

Of the two outer surfaces, one has to be defined as the *upper* or *top* surface, whereas the other is named lower ot *bottom*, thus implicitly orienting the midsurface normal towards the top.

Finally, the body should result fully determined based on a) its midsurface, b) its pointwise thickness, and c) the through-thickness distribution of the constituent materials.

Actually, the geometrical midsurface is of little significance if the material distribution is not symmetric[2]; such midsurface, in fact, exhibits no relevant properties in general. Its definition is nevertheless pretty straighforward.

In the present treatise, a more general *reference* surface definition is preferred to its median geometric counterpart; in particular, an *offset* term is considered that pointwisely shifts the geometric midsurface with respect to the reference surface. A positive offsets shifts the midsurface towards the top.

With the introduction of the offset term, the reference surface may be arbitrarily positioned with respect to the body itself; as an example, an offset set equal to plus or minus half the thickness makes the reference surface correspondent to the bottom or top surfaces, respectively.

Such offset term becomes fundamental in the Finite Element (FE) shell implementation, where, in fact, the reference plane is uniquely defined by the position of the nodes, whereas the offset arbitrarily shifts the geometrical midsurface.

[1] Normal segments may be cast from each point along the midsurface, that end on the outer body surfaces. The midpoint locus of these segments redefines the midsurface itself.

[2] If the unsimmetric laminate is composed by isotropic layers, a reference plane may be obtained for which the $\underline{\underline{B}}$ membrane-to-bending coupling matrix vanishes; a similar condition may not be verified in the presence of orthotropic layers.

In the case of limited[3] curvatures, and for considerations whose scope is local, the tangent reference plane may be employed in place of the possibly curve reference surface, thus locally reducing the general shell treatise to its planar, plate counterpart.

The P displacement components may be defined as a function of the motion of its projection on the reference plane Q. Such Q point is named *reference point* for the through-thickness segment it belongs to.

$$u_P = u + z\,(1+\breve{\epsilon}_z)\sin\varphi \tag{1}$$
$$v_P = v - z\,(1+\breve{\epsilon}_z)\sin\theta \tag{2}$$
$$w_P = w + z\,((1+\breve{\epsilon}_z)\cos(\varphi)\cos(\theta) - 1) \tag{3}$$

The $\breve{\epsilon}_z$ average z strain term is defined based on the accumulation of the Poisson shrinkage (or elongation) along the PQ segment, i.e.

$$\breve{\epsilon}_z(z) = \frac{1}{z}\int_0^z \epsilon_z d\varsigma \tag{4}$$
$$= \frac{1}{z}\int_0^z -\frac{\nu}{1-\nu}\left(\epsilon_x + \epsilon_y\right) d\varsigma \tag{5}$$

The stress component σ_z which is normal to the reference surface is assumed to be either zero or negligible[4].

Such displacement components may be linarized with respect to the small rotations and small ϵ_z strain hypotheses, thus obtaining

[3]with respect to thickness

[4]Such assumption is coherent with the free surface conditions at the top and the bottom skins, and with the moderate thickness of the elastic body, that allows only a narrow deviation from the boundary values.

In fact, the equilibrium of a partitioned, through-thickness material segment requires that

$$\sigma_z(z) = -\int_{-h/2+o}^{z} \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} dz = +\int_{z}^{+h/2-o} \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} dz,$$

where τ_{zx}, τ_{yz} are the interlaminar, out-of-plane shear stress components, whose in-plane gradient is limited.

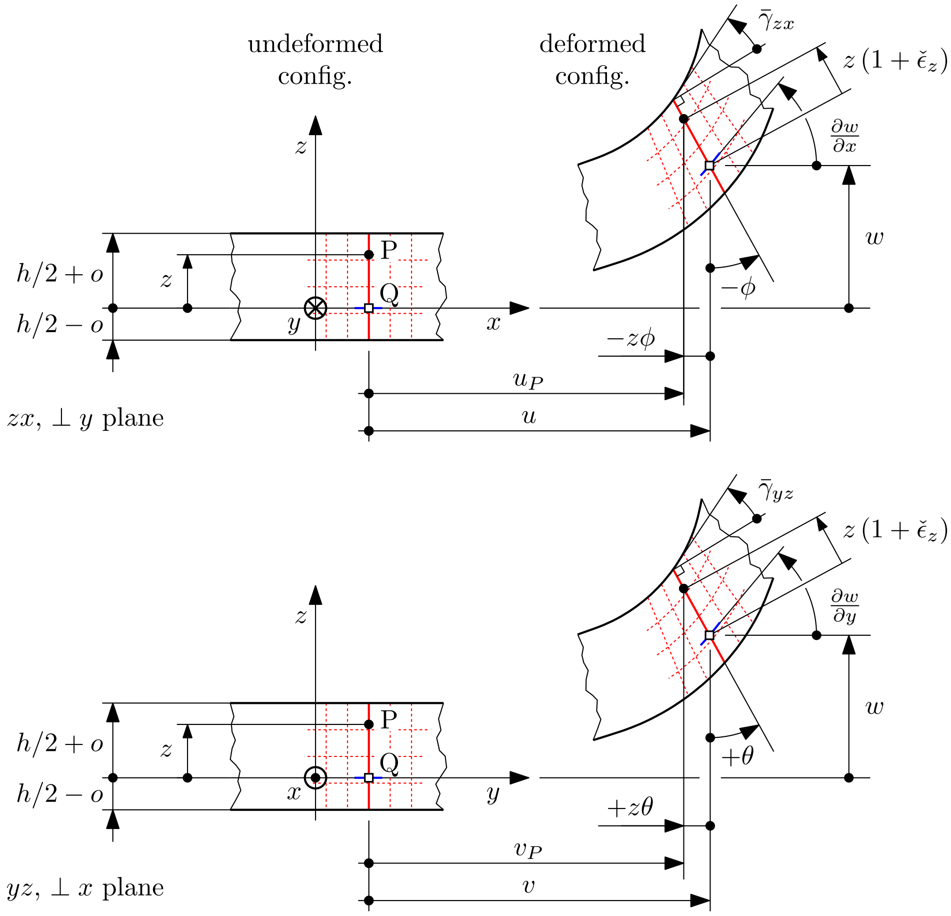

Figure 1: Relevant dimensions for describing the deformable plate kinematics.

$$u_P = u + z\varphi \tag{6}$$
$$v_P = v - z\theta \tag{7}$$
$$w_P = w \tag{8}$$

According to such linearized expression, the kinematic of the P points originally[5] laying on a through-thickness segment that is normal at Q to the reference surface may be described as that of a rigid body. The natural shear related warping is either excluded or neglected, along with the motion of the P points along the segment due to $\breve{\epsilon}_z$.

Also, the behaviour of such a segment is coherent with its rigid body modeling from the external loads point of view; in particular the external actions act on the plate deformable body only through their through-thickness resultants, and no stress/strain components or work are associated by the shell framework to wall squeezing actions, e.g. laminations.

Relation between the normal displacement x, y gradient (i.e. the deformed plate slope), the rotations and the out-of-plane, interlaminar, averaged shear strain components.

$$\frac{\partial w}{\partial x} = \bar{\gamma}_{zx} - \varphi \tag{9}$$
$$\frac{\partial w}{\partial y} = \bar{\gamma}_{yz} + \theta \tag{10}$$

Strains at point P.

$$\epsilon_x = \frac{\partial u_P}{\partial x} = \frac{\partial u}{\partial x} + z\frac{\partial \varphi}{\partial x} \tag{11}$$
$$\epsilon_y = \frac{\partial v_P}{\partial y} = \frac{\partial v}{\partial y} - z\frac{\partial \theta}{\partial y} \tag{12}$$
$$\gamma_{xy} = \frac{\partial u_P}{\partial y} + \frac{\partial v_P}{\partial x} \tag{13}$$
$$= \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + z\left(+\frac{\partial \varphi}{\partial y} - \frac{\partial \theta}{\partial x}\right) \tag{14}$$

[5] i.e. in the undeformed configuration

Generalized plate strains: membrane strains

$$\underline{\bar{\epsilon}} = \left(\begin{array}{c} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{array} \right) = \left(\begin{array}{c} \bar{\epsilon}_x \\ \bar{\epsilon}_y \\ \bar{\gamma}_{xy} \end{array} \right) \tag{15}$$

Generalized plate strains: curvatures.

$$\underline{\kappa} = \left(\begin{array}{c} +\frac{\partial \varphi}{\partial x} \\ -\frac{\partial \theta}{\partial y} \\ +\frac{\partial \varphi}{\partial y} - \frac{\partial \theta}{\partial x} \end{array} \right) = \left(\begin{array}{c} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{array} \right) \tag{16}$$

Compact form for the strain components at P.

$$\underline{\epsilon} = \underline{\bar{\epsilon}} + z\,\underline{\kappa} \tag{17}$$

Hook law for an isotropic material, under plane stress conditions.

$$\underline{\underline{D}} = \frac{E}{1-\nu^2} \left(\begin{array}{ccc} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{array} \right) \tag{18}$$

Normal components for stress and strain, the latter for the isotropic material case only.

$$\sigma_z = 0 \tag{19}$$

$$\epsilon_z = -\frac{\nu}{1-\nu} \left(\epsilon_x + \epsilon_y \right) \tag{20}$$

Stresses at P.

$$\underline{\sigma} = \underline{\underline{D}}\,\underline{\epsilon} = \underline{\underline{D}}\,\underline{\bar{\epsilon}} + z\,\underline{\underline{D}}\,\underline{\kappa} \tag{21}$$

Membrane (direct and shear) stress resultants (stress flows).

$$\underline{q} = \left(\begin{array}{c} q_x \\ q_y \\ q_{xy} \end{array} \right) = \int_h \underline{\sigma}\, dz \tag{22}$$

$$= \underbrace{\int_h \underline{\underline{D}}\, dz}_{\underline{\underline{A}}}\ \underline{\bar{\epsilon}} + \underbrace{\int_h \underline{\underline{D}}\, z dz}_{\underline{\underline{B}}}\ \underline{\kappa} \tag{23}$$

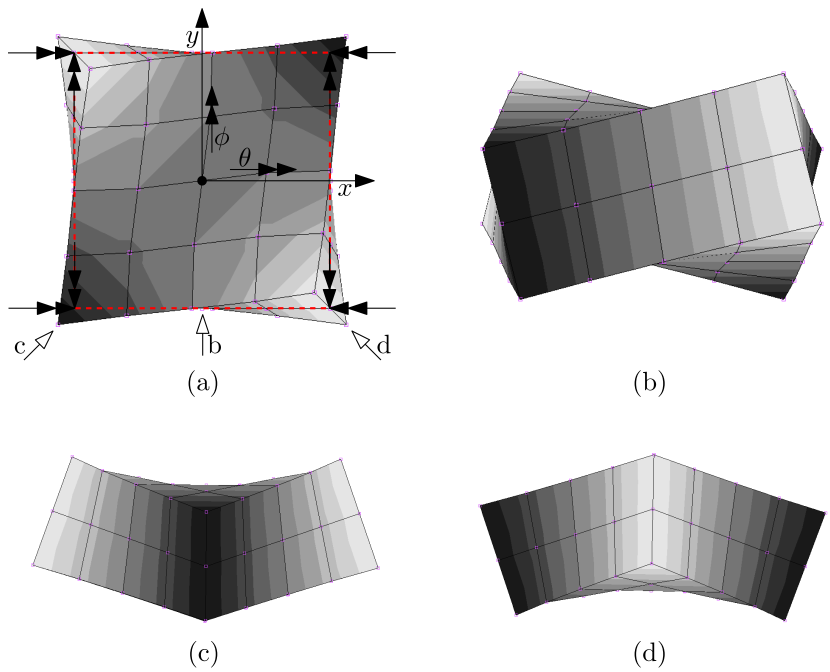

Figure 2: Positive κ_{xy} torsional curvature for the plate element. Subfigure (a) shows the positive γ_{xy} shear strain at the upper surface, the (in-plane) undeformed midsurface, and the negative γ_{xy} at the lower surface; the point of sight related to subfigures (b) to (d) are also evidenced. θ and φ rotation components decrease with x and increase with y, respectively, thus leading to positive κ_{xy} contributions. As shown in subfigures (c) and (d), the torsional curvature of subfigure (b) evolves into two anticlastic bending curvatures if the reference system is aligned with the square plate element diagonals, and hence rotated by 45° with respect to z.

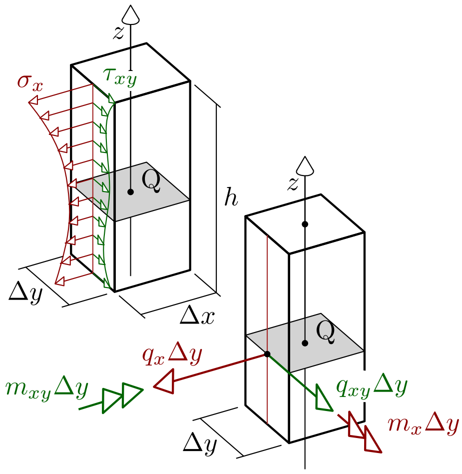

Figure 3: XXX

Bending and torsional moment stress resultants (moment flows).

$$\underline{m} = \begin{pmatrix} m_x \\ m_y \\ m_{xy} \end{pmatrix} = \int_h \underline{\sigma}\, dz \tag{24}$$

$$= \underbrace{\int_h \underline{\underline{D}}\, z dz}_{\underline{\underline{B}} \equiv \underline{\underline{B}}^{\mathrm{T}}}\ \bar{\underline{\epsilon}} + \underbrace{\int_h \underline{\underline{D}}\, z^2 dz}_{\underline{\underline{C}}}\ \underline{\kappa} \tag{25}$$

Cumulative generalized strain - stress relations for the plate (or for the laminate)

$$\begin{pmatrix} \underline{q} \\ \underline{m} \end{pmatrix} = \begin{pmatrix} \underline{\underline{A}} & \underline{\underline{B}} \\ \underline{\underline{B}}^{\mathrm{T}} & \underline{\underline{C}} \end{pmatrix} \begin{pmatrix} \bar{\underline{\epsilon}} \\ \underline{\kappa} \end{pmatrix} \tag{26}$$

Hook law for the orthotropic material in plane stress conditions, with respect to principal axes of orthotropy;

$$\underline{\underline{D}}_{123} = \begin{pmatrix} \frac{E_1}{1-\nu_{12}\nu_{21}} & \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} & 0 \\ \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & \frac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{pmatrix} \tag{27}$$

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{pmatrix} = \underline{\underline{T}}_1 \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} \qquad \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{pmatrix} = \underline{\underline{T}}_2 \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{pmatrix} \tag{28}$$

where

$$\underline{\underline{T}}_1 = \begin{pmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{pmatrix} \tag{29}$$

$$\underline{\underline{T}}_2 = \begin{pmatrix} m^2 & n^2 & mn \\ n^2 & m^2 & -mn \\ -2mn & 2mn & m^2 - n^2 \end{pmatrix} \tag{30}$$

α is the angle between 1 and x;

$$m = \cos(\alpha) \qquad n = \sin(\alpha) \tag{31}$$

The inverse transformations may be obtained based on the relations

$$\underline{\underline{T}}_1^{-1}(+\alpha) = \underline{\underline{T}}_1(-\alpha) \qquad \underline{\underline{T}}_2^{-1}(+\alpha) = \underline{\underline{T}}_2(-\alpha) \tag{32}$$

Finally

$$\underline{\sigma} = \underline{\underline{D}}\,\underline{\epsilon} \qquad \underline{\underline{D}} \equiv \underline{\underline{D}}_{xyz} = \underline{\underline{T}}_1^{-1} \underline{\underline{D}}_{123} \underline{\underline{T}}_2 \tag{33}$$

Notes:

- Midplane is ill-defined if the material distribution is not symmetric; the geometric midplane (i.e. the one obtained by ignoring the material distribution) exhibits no relevant properties in general. Its definition is nevertheless pretty straighforward.

- If the unsimmetric laminate is composed by isotropic layers, a reference plane may be obtained for which the $\underline{\underline{B}}$ membrane-to-bending coupling matrix vanishes; a similar condition may not be verified in the presence of orthotropic layers.

- In the present contribution, the *reference* plane is preferred to the usual geometric midplane for expressing the displacement field, even in the case of homogeneous material or symmetric laminates; in FE shell element implementation, in fact, the reference plane is uniquely defined by the position of the nodes, whereas an offset term may arbitrarily shift the geometrical midsurface.

- Thermally induced distortion is not self-compensated in an unsymmetric laminate even if the temperature is held constant through the thickness.

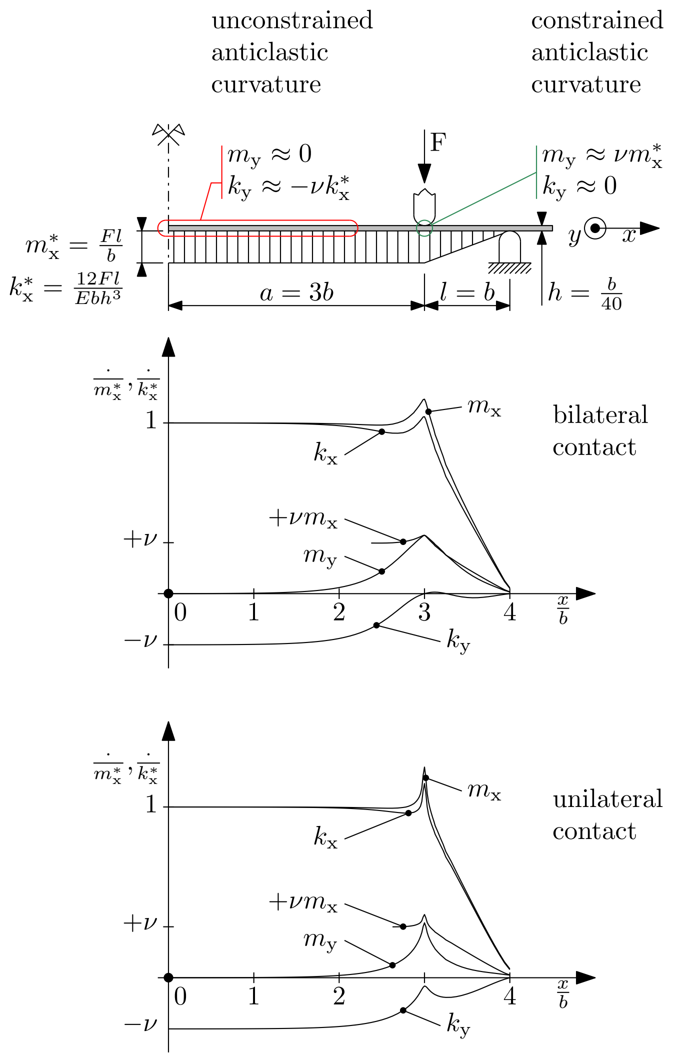

Figure 4: The *not-so-trivial* four point bending case.