Warping Torsion

In addition to shear stresses, some members carry torque by axial stresses. This is called warping torsion. This happens when the cross-section wants to warp, i.e., displace axially, but is prevented from doing so during twisting of the beam. Not all cross-sections warp, and even those that warp do not carry torque by axial stresses unless they are axially restrained at some location(s) along the member. Cross-sections that do NOT warp include axisymmetric cross-sections and thin-walled cross-sections with straight parts that intersect at one point the cross-section, such as X-shaped, T-shaped, and L-shaped cross-sections. For these cross-sections all torque is carried by shear stresses, i.e., St. Venant torsion, regardless of the boundary conditions.

Warping of I-sections

As a pedagogical introduction to warping torsion, consider a beam with an I-section, such as a wide-flange steel beam. When torsion is applied to the beam then the flanges of this cross-section experiences bending in the flange-planes. In other words, torsion induces bending about the strong axis of the flanges. When the flanges are "fixed" at some point, such as in a cantilevered beam with a fully clamped end, some of the torque is carried by axial stresses. To understand this, denote the bending moment and shear force in each flange by M and V, respectively, as shown in Figure 1.

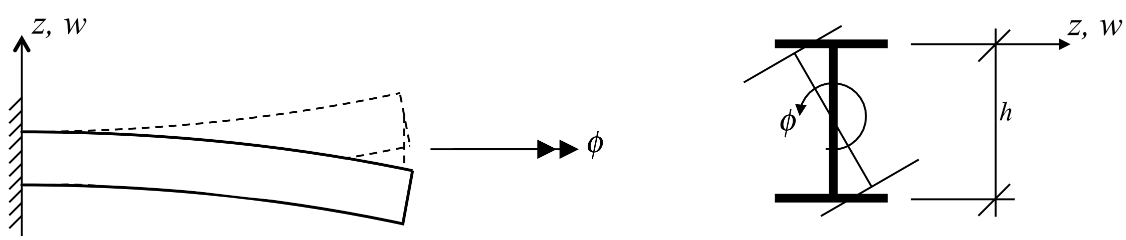

Figure 1: Warping of I-section
(*z* is the local axis for bending of flange, not the global *z*-axis of the cross-section).

The torque that the cross-section carries by bending in the flanges is:

$$T = h \cdot V \tag{1}$$

where h is the distance between the flanges and V is positive shear force in accordance with the document on Euler-Bernoulli beam theory. That document also provides the equilibrium equation that relates shear force to bending moment, which yields:

$$V = \frac{dM}{dx} \quad \Rightarrow \quad T = h \cdot \frac{dM}{dx} \tag{2}$$

The beam theory also provide the relationship between bending moment and flange displacement, w:

$$M = EI \cdot \frac{d^2w}{dx^2} \quad \Rightarrow \quad T = h \cdot EI \cdot \frac{d^3w}{dx^3} \tag{3}$$

where I is the moment of inertia of one flange about its local strong axis. Next, Figure 1 is reviewed to determine the relationship between w and ϕ:

$$w = -\phi \cdot \frac{h}{2} \quad \Rightarrow \quad T = -\frac{h^2}{2} \cdot EI \cdot \frac{d^3\phi}{dx^3} \tag{4}$$

which resulted in the differential equation for warping torsion of an I-section. However, this equation is generally written in this format:

$$T = -EC_w \cdot \frac{d^3\phi}{dx^3} \tag{5}$$

which implies that, for I-sections, the cross-sectional constant for warping is:

$$C_w = I \cdot \frac{h^2}{2} \tag{6}$$

where it is reiterated that I is the moment of inertia of one flange about its local strong axis.

Complete Differential Equation for Torsion

As mentioned earlier, when warping is restrained the torque is carried by both shear stresses, i.e., St. Venant torsion and axial stresses, i.e., warping torsion. Specifically, the torque from shear and axial stresses are superimposed, which leads to the following complete differential equation for torsion:

$$T = GJ \cdot \frac{d\phi}{dx} - EC_w \cdot \frac{d^3\phi}{dx^3} \tag{7}$$

When equilibrium with distributed torque along the beam, m_x, is included, i.e., $m_x=-dT/dx$, then the full differential equation reads

$$EC_w \cdot \frac{d^4\phi}{dx^4} - GJ \cdot \frac{d^2\phi}{dx^2} = m_x \tag{8}$$

Solution

The characteristic equation to obtain the homogeneous solution for the differential equation in Eq. (8) reads

$$\gamma^4 - \frac{GJ}{EC_w} \cdot \gamma^2 = 0 \tag{9}$$

The roots are 0, 0, $\sqrt{(GJ/EC_w)}$, and $-\sqrt{(GJ/EC_w)}$. Accordingly, the homogeneous solution is

$$\phi(x) = C_1 \cdot e^{\sqrt{GJ/EC_w} \cdot x} + C_2 \cdot e^{-\sqrt{GJ/EC_w} \cdot x} + C_3 \cdot x + C_4 \tag{10}$$

which guides the selection of shape functions if an "exact" stiffness matrix with both St. Venant and warping torsion is sought. Another way of expressing the solution is:

$$\phi(x) = C_1 \cdot \sinh\left(\sqrt{GJ/EC_w} \cdot x\right) + C_2 \cdot \cosh\left(\sqrt{GJ/EC_w} \cdot x\right) + C_3 \cdot x + C_4 \tag{11}$$

where the coefficients, C_i, in Eq. (10) are different from those in Eq. (11). For example, the homogeneous solution for a cantilevered beam that is fully fixed at x=0 and subjected to a torque, T_o, at x=L is:

$$\phi(x) = \frac{1}{\sqrt{GJ/EC_w}} \cdot \frac{T_o}{GJ} \cdot \begin{pmatrix} \tanh\left(\sqrt{GJ/EC_w} \cdot L\right) \cdot \left[\cosh\left(\sqrt{GJ/EC_w} \cdot x\right) - 1\right] \\ -\sinh\left(\sqrt{GJ/EC_w} \cdot x\right) + \sqrt{GJ/EC_w} \cdot x \end{pmatrix} \tag{12}$$

From this solution the torque carried by St. Venant torsion is computed by:

$$T_{St.V.}(x) = GJ \cdot \frac{d\phi}{dx} \tag{13}$$

and the torque carried by warping torsion is computed by:

$$T_{warping}(x) = -EC_w \cdot \frac{d^3\phi}{dx^3} \tag{14}$$

where $T_{St.V.}(x)+T_{warping}(x)=T_o$ for all $0<x<L$.

Bi-moment

In the theory of warping torsion the "bi-moment," B, is defined as an auxiliary quantity. This has two primary objectives. The first is to introduce a "degree of freedom" for beam elements that carry torque by restrained warping. The other objective stems from our desire to formulate a theory with a quantity that is tantamount to the ordinary bending moment in beam theory. In other words, the objective is to establish an equation of the form $B=EC_w\phi''$, which is analogous to the equation $M=EIw''$from beam theory. To this end, let the bi-moment for I-sections be defined by

$$B \equiv M \cdot h \tag{15}$$

where M is again the bending moment in the flange about its strong axis. Substitution of the relationship between bending moment in the flange, w, and the flange displacement, w, from Euler-Bernoulli beam theory yields

$$B = EI \cdot \frac{d^2 w}{dx^2} \cdot h \tag{16}$$

and substitution of the relationship between w and ϕ from Eq. (4) yields

$$B = -EI \cdot \frac{d^2\phi}{dx^2} \cdot \frac{h^2}{2} \tag{17}$$

which in light of Eq. (6) is written

$$B = -EC_w \cdot \frac{d^2\phi}{dx^2} \tag{18}$$

This is the desired result, which shows that Eq. (15) is the appropriate definition of the bi-moment for I-sections. It is emphasized that the bi-moment in itself is not measurable, but it serves as a convenient auxiliary quantity in the theory of warping torsion. When warping degrees of freedom are included in beam elements then the bi-moment in the force vector corresponds to the derivative of the rotation, i.e., ϕ', in the displacement vector.

Unified Bending and Torsion of Thin-walled Cross-sections

The following theory, named after Vlasov, is developed for warping torsion of thin-walled cross-sections. Because warping torsion and beam bending are both formulated in terms of axial stresses it is possible to combine the two theories. In fact, the omission of shear deformation in Euler-Bernoulli beam theory is carried over to the warping theory that is presented in the following. It is noted that no theory of warping for general "thick-walled" cross-sections is currently provided in these documents. Although this is a shortcoming, the presented theory is sufficient for many practical applications. This is because many thick-walled cross-section types are difficult to fully restrain axially. In contrast, it is easier to imagine connection designs for thin-walled cross-sections that provide sufficient axial restraint to develop torque due to axial stresses.

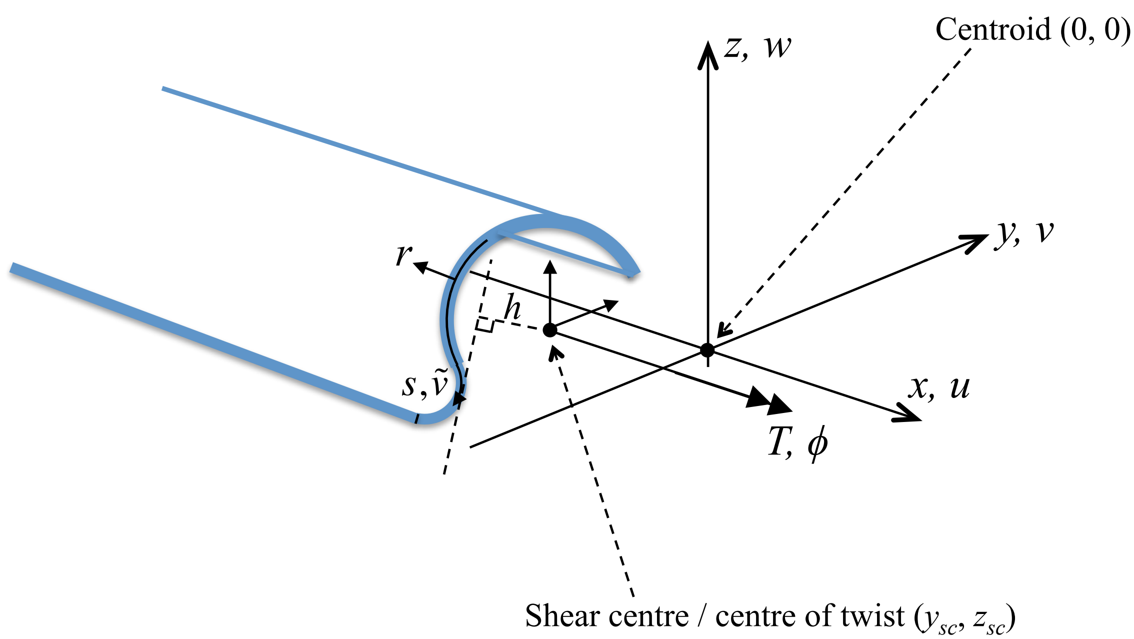

Figure 2: Beam axis system and corresponding displacements.

Kinematics

The objective in this section is to establish a relationship between axial strain, ε_x, and axial displacement, u. For this purpose, let the coordinate s follow the centre line of the contour of the cross-section and let h denote the distance from the centre of rotation (y_{sc}, z_{sc}) to the tangent of the coordinate line s, as shown in Figure 2. The y-z-axes originate in the centroid of the cross-section and the shear centre coordinates y_{sc} and z_{sc} are presumed to be unknown. The part of the deformation that relates to bending is governed by Euler-Bernoulli beam theory, hence the shear strain is neglected. However, for closed cross-sections the shear strain γ_{xs} from St. Venant torsion is included. Specifically, γ_{xs} is equal to the shear strain at r=0 from St. Venant theory, i.e., at the mid-plane of the cross-section profile. For open cross-sections this shear strain is zero. As a fundamental kinematics postulation it is also assumed that the cross-section retains its shape. This implies that $\sigma_s=\varepsilon_s=0$ and that the displacement in the s-direction is:

$$\tilde{v} = -v \cdot \cos(\alpha) + w \cdot \sin(\alpha) + \phi \cdot h \tag{19}$$

where v and w are the displacements of the cross-section and α is the angle between the s-axis and the y-axes. The contributions to Eq. (19) are illustrated in Figure 3. This figure also shows how sin(α) and cos(α) are expressed in terms of the differentials ds, dy, and dz are established, namely:

$$\begin{aligned} \frac{dy}{ds} &= -\cos(\alpha) \\ \frac{dz}{ds} &= \sin(\alpha) \end{aligned} \tag{20}$$

Substitution of Eq. (20) into Eq. (19) yields:

$$\tilde{v} = v \cdot \frac{dy}{ds} + w \cdot \frac{dz}{ds} + \phi \cdot h \tag{21}$$

which expresses that the cross-section retains its shape during deformation.

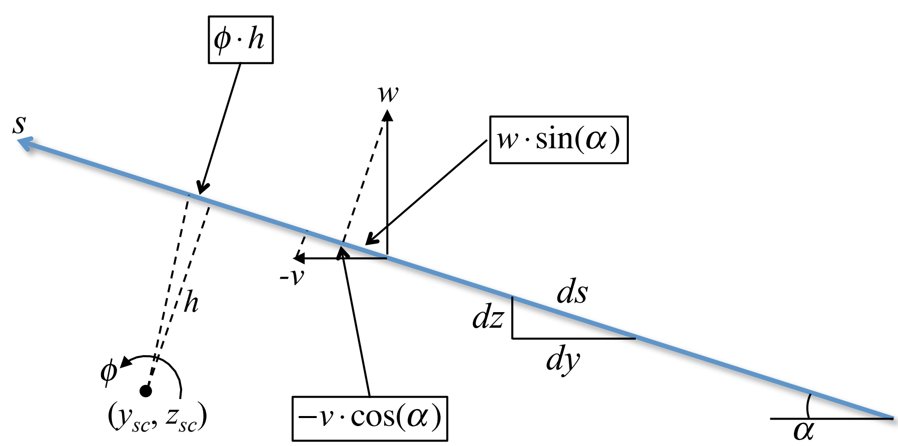

Figure 3: Contributions to the displacement along the *s*-axis.

Because the boundary value problem at hand relates to axial strains, the following equation is the fundamental kinematics equation:

$$\varepsilon_x = \frac{du}{dx} \tag{22}$$

Next, an expression for u is sought, namely the infinitesimal axial displacement, i.e., warping, between two infinitesimally close points in the cross-section. To this end, it is noted that the shear flow in closed cross-sections due to shear force was determined by using the following expressions:

$$\gamma = \frac{du}{ds} \quad \Rightarrow \quad du = \gamma \cdot ds \tag{23}$$

It is also recalled that the derivation of du in St. Venant torsion for closed cross-sections was determined by including rotation of the cross-section:

$$\gamma_{xs} = \frac{d\tilde{v}}{dx} + \frac{du}{ds} \quad \Rightarrow \quad du = \gamma_{xs} \cdot ds - \frac{d\tilde{v}}{dx} \cdot ds = \gamma_{xs} \cdot ds - \frac{d\phi}{dx} \cdot h \cdot ds \tag{24}$$

where the last term represents the axial displacement, i.e., warping, due to the rotation, ϕ. The kinematics of Eq. (24) is also utilized in the following, but the expression for $\tilde{v}$ includes all the terms in Eq. (21). Substitution of Eq. (21) into Eq. (24) yields:

$$du = -\frac{\mathrm{d}v}{\mathrm{d}x} \cdot dy - \frac{\mathrm{d}w}{\mathrm{d}x} \cdot dz - \left(\frac{\mathrm{d}\phi}{\mathrm{d}x} \cdot h - \gamma_{xs} \right) \cdot ds \tag{25}$$

where it is reiterated that γ_{xs} is the shear strain due to St. Venant torsion, which is non-zero only for closed cross-sections. It is desirable to express γ_{xs} in terms of $d\phi/dx$ so that the latter can be pulled outside the parenthesis. This is achieved by utilizing equations from St. Venant theory. First, material law yields

$$\gamma_{xs} = \frac{\tau_{xs}}{G} \tag{26}$$

Furthermore, τ_{xs} equals $\varphi_{,r}$ where φ is Prandtl's stress function, which is cross-section dependent. For thin-walled cross-sections with one cell a good stress function is

$$\varphi(s,r) = K \cdot \left(\frac{1}{2} + \frac{r}{t} \right) \tag{27}$$

As a result, the shear stress is

$$\tau_{xs} = \varphi_{,r} = \frac{K}{t} \tag{28}$$

To express the stress in terms of $d\phi/dx$ it is noted that

$$T = GJ \frac{d\phi}{dx} \tag{29}$$

and that another expression for the torque, T, is available from the stress-resultant equation, which for the stress function in Eq. (27) yields

$$T = 2 \cdot \int_A \varphi \, dA = 2 \cdot K \cdot A_m \tag{30}$$

Solving for K in Eq. (30) and substituting it into Eq. (28), followed by substitution of Eq. (29) yields:

$$\tau_{xs} = \frac{K}{t} = \frac{T}{t \cdot 2 \cdot A_m} = \frac{GJ}{2 \cdot t \cdot A_m} \cdot \frac{d\phi}{dx} \tag{31}$$

Substitution of Eq. (31) into Eq. (25) yields

$$du = -\frac{\mathrm{d}v}{\mathrm{d}x} \cdot dy - \frac{\mathrm{d}w}{\mathrm{d}x} \cdot dz - \frac{d\phi}{dx} \cdot \left(h - \frac{J}{2 \cdot t \cdot A_m} \right) \cdot ds \tag{32}$$

The u-displacement at any point in the cross-section is obtained by summing the infinitesimal contributions in Eq. (32). In other words, integration along the y, z, and s directions yields the complete expression for axial displacement at a point in the cross-section:

$$u(y,z) = u_o - \frac{\mathrm{d}v}{\mathrm{d}x} \cdot y - \frac{\mathrm{d}w}{\mathrm{d}x} \cdot z - \frac{\mathrm{d}\phi}{\mathrm{d}x} \cdot \Omega \tag{33}$$

where u_o is the integration constant, i.e., the axial displacement at the neutral axis, and Ω has been defined as:

$$\Omega \equiv \int \left(h - \frac{J}{2 \cdot t \cdot A_m} \right) ds \tag{34}$$

where it is reemphasized that this expression is valid for cross-sections with one cell. The last term in the integrand is called the “shear radius:”

$$\bar{h} = \frac{J}{2 \cdot t \cdot A_m} \tag{35}$$

which leads to the short-hand notation:

$$\Omega \equiv \int \left(h - \bar{h} \right) ds \tag{36}$$

For open cross-sections the shear strain in the mid-plane of the cross-section profile is zero, which implies that

$$\Omega \equiv \int h \, ds \tag{37}$$

Finally, combining Eq. (22) and Eq. (33) yields the final kinematics equation:

$$\varepsilon_x = \frac{\mathrm{d}u}{\mathrm{d}x} - \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} \cdot y - \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \cdot z - \frac{\mathrm{d}^2 \phi}{\mathrm{d}x^2} \cdot \Omega \tag{38}$$

where u_o has been renamed to u to match the typical notation for truss members.

Material Law

Hooke's law provides the relationship between axial stress and axial strain:

$$\sigma_x = E \cdot \varepsilon_x \tag{39}$$

Section Integration

Integration of axial stress over the cross-section yields the axial force:

$$N = \int_A \sigma_x \, dA \tag{40}$$

Integration of axial stress multiplied by distance from the centroid yields bending moment:

$$M_z = -\int_A \sigma_x \cdot y \, dA \tag{41}$$

Integration of axial stress multiplied by distance from the centroid yields bending moment:

$$M_y = -\int_A \sigma_x \cdot z \, dA \tag{42}$$

Integration of axial stress multiplied by the previously defined quantity Ω is defined as the "bi-moment:"

$$B \equiv -\int_A \sigma_x \cdot \Omega \, dA \tag{43}$$

Equilibrium

Distributed axial load along the beam is related to the axial force by the following equilibrium equation:

$$q_x = -\frac{dN}{dx} \tag{44}$$

Distributed load in *z*-direction, which is assumed to act through the shear centre or it will contribute to m_x, is related to the shear force by the following equilibrium equation:

$$q_z = -\frac{dV_z}{dx} \tag{45}$$

Equilibrium also provides the relationship between shear force and bending moment:

$$V_z = \frac{dM_y}{dx} \tag{46}$$

The corresponding equilibrium equations in the other direction are:

$$q_y = \frac{dV_y}{dx} \tag{47}$$

and

$$V_y = \frac{dM_z}{dx} \tag{48}$$

Finally, equilibrium for distributed torque along the beam yields:

$$m_x = -\frac{dT}{dx} \tag{49}$$

Differential Equations

Substitution of the kinematics equation in Eq. (38) into the material law in Eq. (39) yields:

$$\sigma_x = E \cdot \frac{\mathrm{d}u}{\mathrm{d}x} - E \cdot \frac{\mathrm{d}^2 v}{\mathrm{d}x^2} \cdot y - E \cdot \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \cdot z - E \cdot \frac{\mathrm{d}^2 \phi}{\mathrm{d}x^2} \cdot \Omega \tag{50}$$

Substitution of Eq. (50) into the section integration Eqs. (40) to (43) yields the following set of equations (Weberg 1970):

$$\begin{Bmatrix} N \\ M_z \\ M_y \\ B \end{Bmatrix} = E \cdot \begin{bmatrix} \int_A dA & -\int_A y\,dA & -\int_A z\,dA & -\int_A \Omega\,dA \\ -\int_A y\,dA & \int_A y^2\,dA & \int_A y \cdot z\,dA & \int_A y \cdot \Omega\,dA \\ -\int_A z\,dA & \int_A y \cdot z\,dA & \int_A z^2\,dA & \int_A z \cdot \Omega\,dA \\ -\int_A \Omega\,dA & \int_A y \cdot \Omega\,dA & \int_A z \cdot \Omega\,dA & \int_A \Omega^2\,dA \end{bmatrix} \begin{Bmatrix} \frac{du}{dx} \\ \frac{d^2 w}{dx^2} \\ \frac{d^2 v}{dx^2} \\ \frac{d^2 \phi}{dx^2} \end{Bmatrix} \tag{51}$$

where symmetry is observed. Under certain condition described shortly, the equations become decoupled and reduces to:

$$N = EA \cdot \frac{du}{dx} \tag{52}$$

$$M_y = EI_y \frac{d^2 w}{dx^2} \tag{53}$$

$$M_z = EI_z \frac{d^2 v}{dx^2} \tag{54}$$

$$B = -EC_w \frac{d^2 \phi}{dx^2} \tag{55}$$

where the diagonal components of the matrix in Eq. (51) have been named as follows:

$$A = \int_A \mathrm{d}A \tag{56}$$

$$I_z = \int_A y^2 \, dA \tag{57}$$

$$I_y = \int_A z^2 \, dA \tag{58}$$

$$C_w = \int_A \Omega^2 \, dA \tag{59}$$

For the system of equations in Eq. (51) to be decoupled, the six off-diagonal elements of the coefficient matrix must be zero. These six conditions form an important part of the cross-section analysis. In fact, they determine the following six unknowns of the cross-section:

1. y_o = y-coordinate of the centroid
2. z_o = z-coordinate of the centroid
3. θ_o = orientation of the principal axes
4. C = normalizing constant for the Ω-diagram
5. y_{sc} = y-coordinate of the shear centre
6. z_{sc} = z-coordinate of the shear centre

Specifically, the coordinates of the centroid of the cross-section are determined by:

$$\int_A y \ \mathrm{d}A = \int_A z \ \mathrm{d}A = 0 \tag{60}$$

The orientation of the principal axes are determined by:

$$\int_A y \cdot z \ \mathrm{d}A = 0 \tag{61}$$

The normalizing constant for the Ω-diagram is determined by:

$$\int_A \Omega \ \mathrm{d}A = 0 \tag{62}$$

The shear centre coordinates are determined by:

$$\int_A y \cdot \Omega \ \mathrm{d}A = \int_A z \cdot \Omega \ \mathrm{d}A = 0 \tag{63}$$

Adding equilibrium with external forces yields the final differential equations for axial deformation and bending

$$q_x = -EA \cdot \frac{d^2 u_o}{dx^2} \tag{64}$$

$$q_z = EI_y \frac{d^4 w_D}{dx^4} \tag{65}$$

$$q_y = EI_z \frac{d^4 v_D}{dx^4} \tag{66}$$

The derivation of the differential equation that combines St. Venant torsion and warping torsion starts with the definition of the stress resultant:

$$T = \int_A \tau_{xs} \cdot t \cdot h\, dA = \int_A q_s \cdot h\, ds = \int_A q_s\, d\Omega = \left[q_s \cdot \Omega\right]_\Gamma - \int_A \Omega\, dq_s \tag{67}$$

where q_s is the shear flow and the boundary term $[q_s\,\Omega]_\Gamma$ from integration by parts is zero. Because shear strains are omitted from the warping theory it is necessary to employ equilibrium to recover the shear flow. With reference to Figure 4, equilibrium yields:

$$d\sigma_x \cdot ds \cdot t + d\tau_{xs} \cdot dx \cdot t = 0 \quad \Rightarrow \quad \frac{d\sigma_x}{dx} \cdot t + \frac{d\tau_{xs}}{ds} \cdot t = 0 \quad \Rightarrow \quad \frac{dq_s}{ds} = -\frac{d\sigma_x}{dx} \cdot t \tag{68}$$

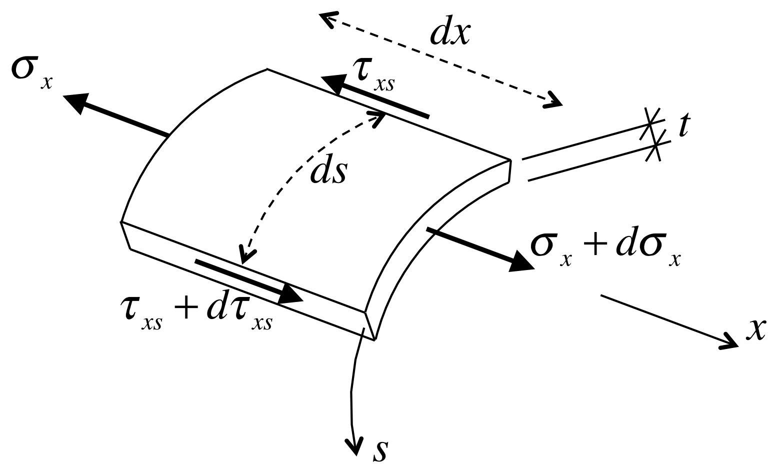

Figure 4: Equilibrium to recover shear stresses.

Substitution of Eq. (68) into Eq. (67) yields:

$$T = -\int_A \Omega\, dq_s = \int_A \Omega \cdot \frac{d\sigma_x}{dx} \cdot t\, ds = \int_A \Omega \cdot \frac{d\sigma_x}{dx} dA = \frac{d}{dx} \int_A \Omega \cdot \sigma_x\, dA = -\frac{dB}{dx} \tag{69}$$

Adding the torque carried by shear stresses, i.e., $T=GJ(d\phi/dx)$ and employing Eq. (55) yields:

$$T = GJ \cdot \frac{d\phi}{dx} - EC_w \frac{d^3\phi}{dx^3} \tag{70}$$

Adding equilibrium with distributed torque from Eq. (49) yields the complete differential equation for St. Venant torsion and warping torsion:

$$m_x = EC_w \frac{d^4\phi}{dx^4} - GJ \cdot \frac{d^2\phi}{dx^2} \tag{71}$$

The solution to this differential equation was presented in Eq. (10).

Modified Theory for Closed Cross-sections

The theory presented above is now modified with a new expression *u*, i.e., with a new formulation of the warping (Hals 1993). The focus remains on thin-walled cross-sections, and the correction is particularly aimed at improving the results for closed cross-sections. A key characteristic of the modified theory is that both shear and axial stresses are considered in the same boundary value problem. This is new, because only axial strains and stresses are considered in Euler-Bernoulli beam theory and the warping theory above. Conversely, the St. Venant warping theory is formulated in terms of shear stresses. Figure 5 is included to emphasize the combined boundary value problem that is now considered.

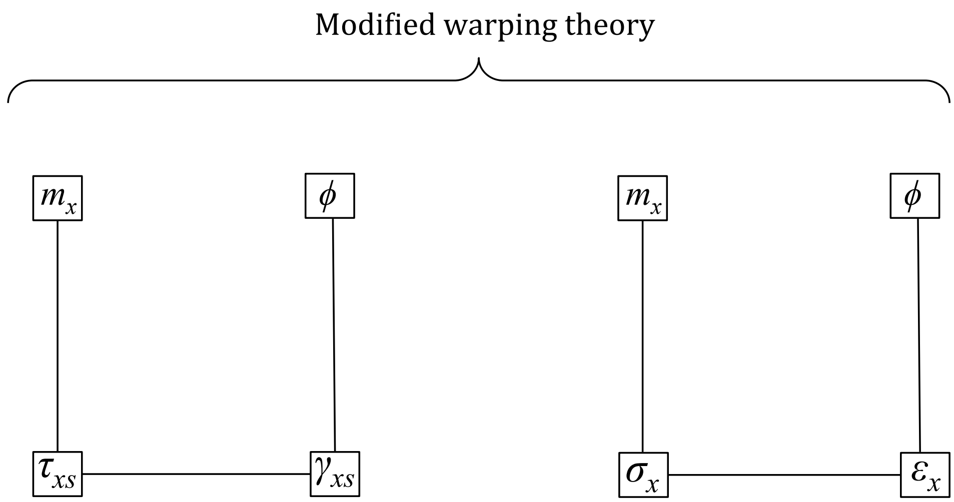

Figure 5: Combined BVP for warping torsion considering both shear and axial strains.

The key modification in this theory is a revision of Eq. (33), while Eq. (19) is maintained. The warping due to rotation of the cross-section is now written

$$u(y,z) = -F(x)\cdot\Omega \tag{72}$$

instead of

$$u(y,z) = -\frac{\mathrm{d}\phi}{\mathrm{d}x}\cdot\Omega \tag{73}$$

The function $F(x)$ is so far unknown and generally different from ϕ'.

Boundary Value Problem for Shear

Kinematic considerations yield the shear strain:

$$\gamma_{xs} = \frac{d\tilde{v}}{dx} + \frac{du}{ds} = \frac{d\phi}{dx}\cdot h - F\cdot\frac{d\Omega}{ds} \tag{74}$$

Material law added to the kinematics equation yields:

$$\tau_{xs} = G\cdot\left(\frac{d\phi}{dx}\cdot h - F\cdot\frac{d\Omega}{ds}\right) \tag{75}$$

Section integration of shear stresses around the cell yields the total torque:

$$\begin{aligned} T &= \oint \tau_{xs} \cdot t \cdot h \cdot ds = \oint G \cdot \left(\frac{d\phi}{dx} \cdot h - F \cdot \frac{d\Omega}{ds} \right) \cdot t \cdot h \cdot ds \\ &= G \cdot \frac{d\phi}{dx} \cdot \oint t \cdot h^2 \cdot ds - G \cdot F \cdot \oint \frac{d\Omega}{ds} \cdot t \cdot h \cdot ds \end{aligned} \tag{76}$$

This equation can be simplified. First, the following definition is made:

$$J_h = \oint t \cdot h^2 \cdot ds = \int_A h^2 \cdot dA \tag{77}$$

Second, in accordance with Eq. (36) it is recognized that

$$\frac{d\Omega}{ds} = \left(h - \bar{h} \right) \tag{78}$$

This means that Eq. (76) takes the form

$$T = G \cdot \frac{d\phi}{dx} \cdot J_h - G \cdot F \cdot \oint h^2 \cdot t \cdot ds + G \cdot F \cdot \oint \bar{h} \cdot h \cdot t \cdot ds \tag{79}$$

Interestingly, the last term can be rewritten in terms of the cross-sectional constant for St. Venant warping. Introducing Eq. (35) yields:

$$\oint h \cdot \bar{h} \cdot t \cdot ds = \oint h \cdot \left(\frac{J}{2 \cdot t \cdot A_m} \right) \cdot t \cdot ds = \frac{J}{2 \cdot A_m} \cdot \underbrace{\oint h \cdot ds}_{2 \cdot A_m} = J \tag{80}$$

Thus, Eq. (79) turns into:

$$T = G \cdot J_h \cdot \frac{d\phi}{dx} - G \cdot F \cdot (J_h - J) \tag{81}$$

If the cross-section has protruding flanges then those are added according to the basic St. Venant formula T=GJϕ':

$$T = G \cdot \frac{d\phi}{dx} \cdot (J_h + J_{flanges}) - G \cdot F \cdot (J_h - J) \tag{82}$$

Finally, after having employed kinematics, material law, and section integration, equilibrium with applied distributed torque is added in accordance with Eq. (49), which substituted into Eq. (82) yields the differential equation:

$$G \cdot (J_h + J_{flanges}) \cdot \frac{d^2\phi}{dx^2} - G \cdot (J_h - J) \cdot \frac{dF}{dx} = -m_x \tag{83}$$

Boundary Value Problem for Axial

Kinematic considerations without bending and truss action yield the axial strain:

$$\varepsilon_x = -\frac{dF}{dx} \cdot \Omega \tag{84}$$

Material law added to the kinematic equation yields:

$$\sigma_x = -E \cdot \frac{dF}{dx} \cdot \Omega \tag{85}$$

Point-wise equilibrium in solid mechanics is expressed in index notation as $\sigma_{ij,i}+p_j=0$. This leads to the following equilibrium equation for points on the cross-section contour:

$$t \cdot \sigma_{x,x} + t \cdot \tau_{xs,s} + p_x = 0 \tag{86}$$

where p_x is the force-intensity in the x-direction at that point. Only the weak form of this equilibrium equation is employed in this theory. For this purpose, Eq. (86) is weighted and integrated over the cross-section:

$$\begin{aligned}\int \left(t \cdot \sigma_{x,x} + t \cdot \tau_{xs,s} + p_x\right) \cdot \Omega(s) \cdot ds = &\int \sigma_{x,x} \cdot \Omega(s) \cdot t \cdot ds \\ &+\int \tau_{xs,s} \cdot \Omega(s) \cdot t \cdot ds \\ &+\int p_x \cdot \Omega(s) \cdot ds \\ &= 0\end{aligned} \tag{87}$$

where the weight function is the cross-sectional warping, represented by Ω. Each of the three terms in Eq. (87) is further developed in the following. The first term is modified by substitution of Eq. (85):

$$\int \sigma_{x,x} \cdot \Omega \cdot t \cdot ds = -E \cdot \frac{d^2F}{dx^2} \cdot \int \Omega^2 \cdot t \cdot ds = -EC_w \cdot \frac{d^2F}{dx^2} \tag{88}$$

The second term in Eq. (87) is rewritten by integration by parts:

$$\int \frac{d\tau_{xs}}{ds} \cdot t \cdot \Omega(s) \cdot ds = \left[\tau_{xs} \cdot t \cdot \Omega(s)\right] - \int \tau_{xs} \cdot \frac{d\Omega(s)}{ds} \cdot t \cdot ds \tag{89}$$

where the boundary term cancels. Eq. (89) is further rewritten by substitution of Eq. (78):

$$\int \tau_{xs} \cdot \frac{d\Omega(s)}{ds} \cdot t \cdot ds = \int \tau_{xs} \cdot (h - \overline{h}) \cdot t \cdot ds \tag{90}$$

This is expanded by substitution of Eq. (75):

$$\int \tau_{xs} \cdot (h-\overline{h}) \cdot t \cdot ds = G \cdot \phi' \cdot \int (h^2 - h\overline{h}) \cdot t \cdot ds - G \cdot F \cdot \int (h-\overline{h})^2 \cdot t \cdot ds \tag{91}$$

Introducing cross-section constants that are defined earlier, including the result from Eq. (80), yields:

$$\begin{aligned}&G \cdot \phi' \cdot \int (h^2 - h\overline{h}) \cdot t \cdot ds - G \cdot F \cdot \int (h-\overline{h})^2 \cdot t \cdot ds \\ &= G \cdot \phi' \cdot \left(J_h - J\right) - G \cdot F \cdot \left(J_h - 2 \cdot J + \int \overline{h}^2 \cdot t \cdot ds\right)\end{aligned} \tag{92}$$

This can be further simplified because Bredt's formula from St. Venant torsion yields

$$\int \overline{h}^2 \cdot t \cdot ds = \int \left(\frac{J}{2 \cdot t \cdot A_m}\right)^2 \cdot t \cdot ds = J^2 \cdot \frac{\int \frac{1}{t} ds}{4 \cdot A_m^2} = J \tag{93}$$

The third term in Eq. (87) defines the "warping load" on the cross-section:

$$m_\Omega = \int p_x \cdot \Omega(s) \cdot ds \tag{94}$$

In summary, Eq. (87) is written as the following differential equation:

$$-EC_w \cdot F'' - G \cdot \left(J_h - J\right) \cdot \phi' + G \cdot \left(J_h - J\right) \cdot F + m_\Omega = 0 \tag{95}$$

Combined Differential Equation

The differential equation for the shear-BVP in Eq. (83) and the differential equation for the axial-BVP in Eq. (95) are now combined. First Eq. (83) is solved for *F'*:

$$F' = \left(\frac{J_h}{(J_h - J)} + \frac{J_{flanges}}{(J_h - J)} \right) \cdot \phi'' + \frac{m_x}{G \cdot (J_h - J)} \tag{96}$$

By differentiating twice, this equation is also employed to obtain an expression for *F'''*:

$$F''' = \left(\frac{J_h}{(J_h - J)} + \frac{J_{flanges}}{(J_h - J)} \right) \cdot \phi'''' + \frac{m_x''}{G \cdot (J_h - J)} \tag{97}$$

The next step is to differentiate Eq. (95) once with respect to x and applying a minus-sign to it:

$$EC_w \cdot F''' + G \cdot \left(J_h - J\right) \cdot \phi'' - G \cdot \left(J_h - J\right) \cdot F' - m_\Omega' = 0 \tag{98}$$

Substitution of Eqs. (96) and (97) into Eq. (98) yields

$$\begin{aligned} & EC_w \cdot \left(\left(\frac{J_h}{(J_h - J)} + \frac{J_{flanges}}{(J_h - J)} \right) \cdot \phi'''' + \frac{m_x''}{G \cdot (J_h - J)} \right) \\ & + G \cdot \left(J_h - J\right) \cdot \phi'' - G \cdot \left(J_h - J\right) \cdot \left(\left(\frac{J_h}{(J_h - J)} + \frac{J_{flanges}}{(J_h - J)} \right) \cdot \phi'' + \frac{m_x}{G \cdot (J_h - J)} \right) \\ & - m_\Omega' = 0 \end{aligned} \tag{99}$$

By re-arranging and defining the following auxiliary constants:

$$\alpha_o = \frac{J_h}{J_h - J} \tag{100}$$

$$\beta_o = \frac{J_{flanges}}{J_h} \tag{101}$$

$$\kappa_o = \frac{J_{flanges}}{J} \tag{102}$$

the following complete differential equation that contains both the axial-BVP and the shear-BVP (Hals 1993) is obtained:

$$EC_w \cdot \alpha_o \left(1 + \beta_o\right) \cdot \phi'''' - GJ \cdot \left(1 + \kappa_o\right) \cdot \phi'' = m_\Omega' + m_x - \frac{EC_w}{GJ_h} \cdot \alpha_o \cdot m_x'' \tag{103}$$

Problems without free flanges are characterized by β_o=κ_o=0 and the homogeneous differential equation for such problems is:

$$EC_w \cdot \alpha_o \cdot \phi'''' - GJ \cdot \phi'' = 0 \tag{104}$$

which has the general solution:

$$\phi = C_1 + C_2 \cdot x + C_3 \cdot e^{k_o x} - C_4 \cdot e^{-k_o x} \tag{105}$$

where

$$k_o = \frac{GJ}{EC_w \alpha_o} \tag{106}$$

Once a solution to the differential equation is obtained, the unknown function $F(x)$ can be determined. Solving Eq. (95) yields

$$F = -\frac{m_\Omega}{G \cdot (J_h - J)} + \frac{EC_w \cdot F''}{G \cdot (J_h - J)} + \phi' \tag{107}$$

and substituting F'' from Eq. (96) differentiated once yields, when there are no free flanges:

$$F = -\frac{m_\Omega}{GJ_h} \cdot \alpha_o + \frac{EC_w \cdot \alpha_o^2}{GJ_h} \cdot \phi''' + \frac{EC_w \alpha_o}{GJ_h} \cdot \frac{m_x'}{GJ_h} \cdot \alpha_o + \phi' \tag{108}$$

This expression reveals the warping of the cross-section, but it is not needed to determine the bi-moment for axial stress computations. This is understood by first considering the definition of the bi-moment, which is:

$$B = -\int_A \sigma_x \cdot \Omega \, dA \tag{109}$$

Substitution of the expression for axial stress from Eq. (85) yields

$$B = \int_A E \cdot \frac{dF}{dx} \cdot \Omega^2 \, dA = EC_w \cdot F' \tag{110}$$

Instead of employing Eq. (108) to determine *F'*, it is possible to substitute F' from Eq. (96), which yields, when there are no free flanges:

$$B = EC_w \cdot \alpha_o \cdot \phi'' + \frac{EC_w}{GJ_h} \cdot \alpha_o \cdot m_x \tag{111}$$

The axial stress in the cross-section is obtained by combining Eq. (110) with Eq. (85):

$$\sigma_x = -\frac{B}{C_w} \cdot \Omega \tag{112}$$

The total torque is obtained by combining the expression for torque in Eq. (97) with the differential equation for shear in Eq. (83) and the differential equation for axial in Eq. (95). Substitution of *F* from Eq. (95) and then *F''* from the differentiated Eq. (83) into Eq. (82) yields:

$$T = -EC_w \cdot \alpha_o \cdot \phi''' + GJ \cdot \phi' - \frac{EC_w \cdot \alpha_o}{GJ_h} \cdot m_x' + m_\Omega \tag{113}$$

The shear stresses are obtained from Eq. (75):

$$\tau_{xs} = G \cdot \phi' \cdot h - G \cdot F \cdot (h - \bar{h}) \tag{114}$$

where the expression for F from Eq. (108) could conceivably be utilized. However, a better way of determining the stress stresses is to solve the statically indeterminate shear flow around the cell by enforcing compatibility.

References

Hals, T. E. (1993). *Tynnveggede staver*. Tapir.

Weberg, S. E. (1970). *Torsion og bøyning: massive, åpne og lukkede tverssnitt*. Institutt for statikk, NTH.