UNIMORE
UNIVERSITÀ DEGLI STUDI DI MODENA E REGGIO EMILIA

Dipartimento di Ingegneria
"Enzo Ferrari"

Progettazione Assistitia di Organi di Macchine

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Agenda

Castigliano's Theorem analysis of structures by Maxima:

- CASE D: Rollbar
- Centroid and Shear center
- Symmetry BCs
- Skew-symmetry BCs
- Symm and skew-symmetry BCs
- References

Agenda

Castigliano's Theorem analysis of a rollbar (Case D) by Maxima

- Case D: Rollbar solution
- Consideration: case 1: b>>>a
- Consideration: case 2: b<<<a
- Consideration: case 3: b=0
- Influence of the beam aspect ratio

Centroid and Shear center

Symmetry BCs

Skew-symmetry BCs

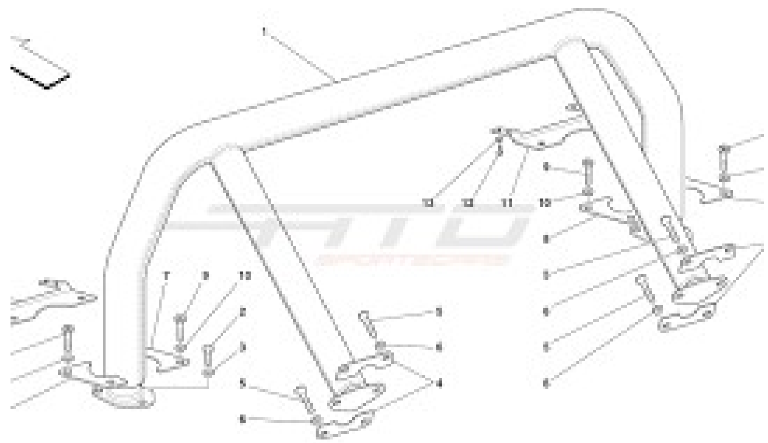
Symm and skew-symmetry BCs

Modelling technique

References

Castigliano's Theorem

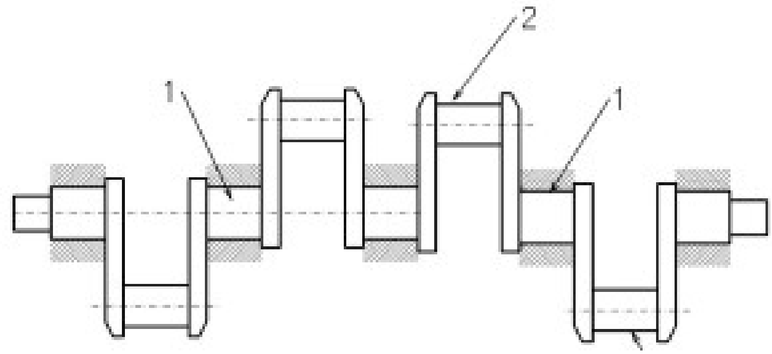
CASE D: Roll bar - Statically redundant structure (+3 dof)

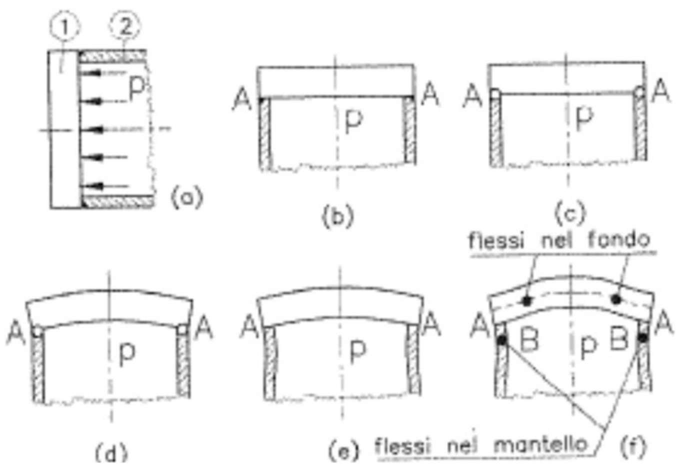


Castigliano’s Theorem

Rollbar and further mechanical applications

Crankshaft throw

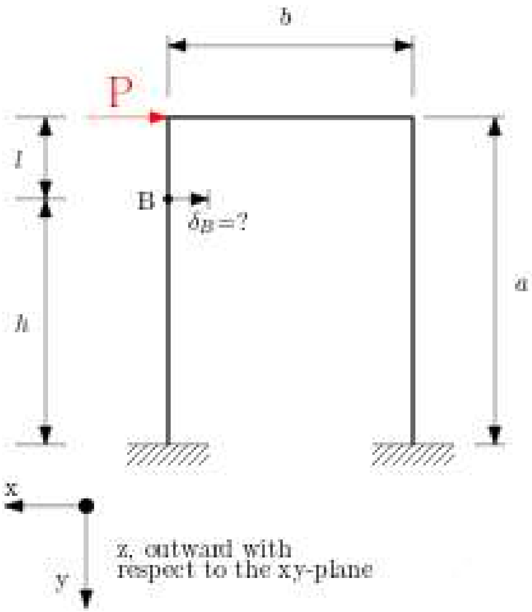




A pressurized vessel where the cylinder tube and the cap are welded (axisymmetric geometry).
Approximate to a plane frame omitting the pressure acting at the cylinder tube.

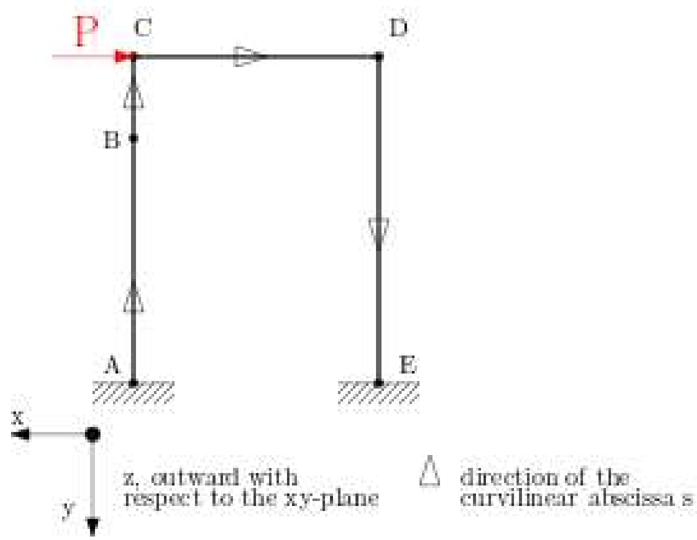
Castigliano's Theorem

CASE D: Roll bar - solution



Considering a simplified roll bar:
- fixed to the extremities;
- loaded by a lateral concentrated force (P) acting at the point B of the structure.

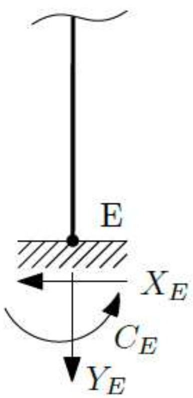
Evaluate the deflection (δ_B) acting at the point B of the structure, located at the maximum point at which the driver and the passenger can reach during a rollover crash event.

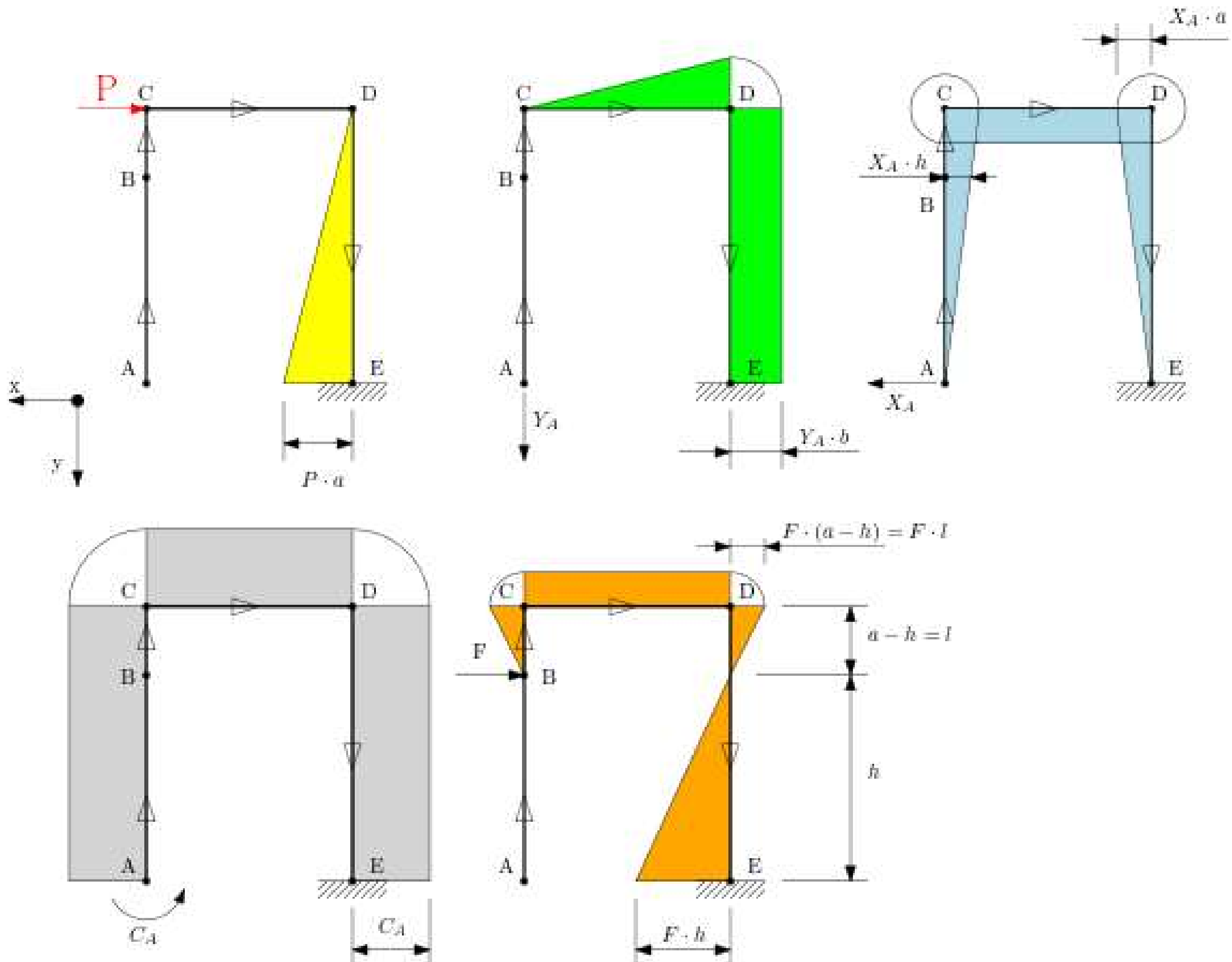


Castigliano's Theorem

CASE D: Roll bar - solution

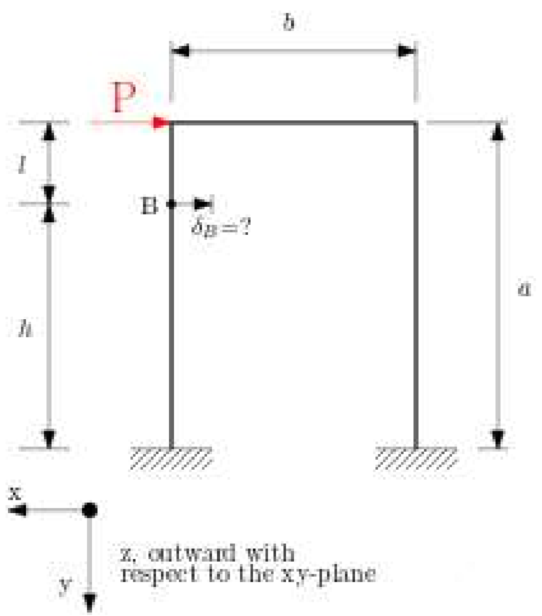
Reaction force and moment acting at point E are assumed positive, and they are named as follows.





Castigliano’s Theorem

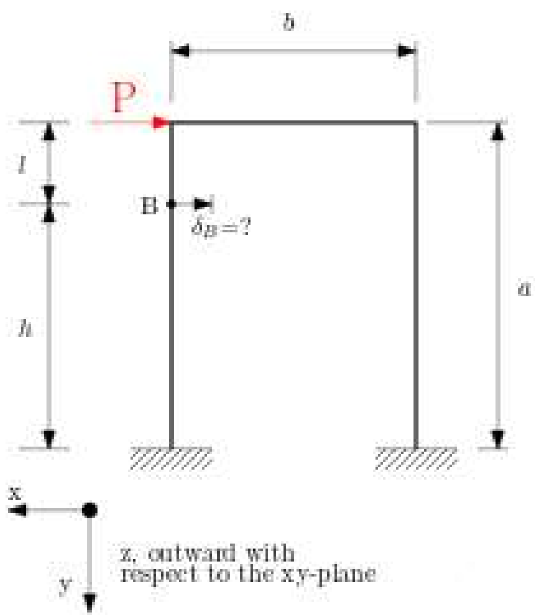
CASE D: Roll bar - solution



1. Application of a fictitious force *F* at point B;
2. Evaluation of the equilibrium of the structure, to retrieve the reaction forces and moment acting at E;
3. Definition of a linear shape functions in the [0,1] interval;
4. Definition of the bending moment acting on the portions of the structure, called as `Mf_AB, Mf_BC, Mf_CD, Mf_DE`;
5. Definition of the elastic internal energy related to the various beam segments `U_AB, U_BC, U_CD, U_DE`;
6. Evaluation of the total elastic internal energy of the structure defined as the sum of the various beam segments.

Castigliano's Theorem

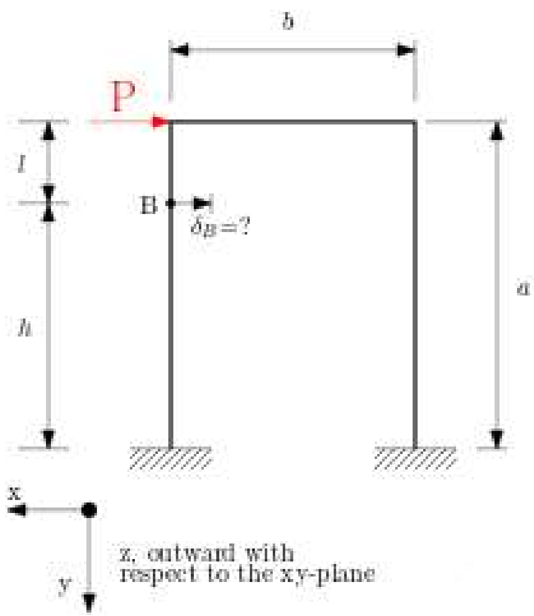
CASE D: Roll bar - solution



7. Application of the Castigliano's theorem for obtaining the displacements and the rotation at A named as `uA, vA, rA.`
8. Definition of kinematic congruence with respect to the clamp constraint in A is to be enforced, by the mean of a system of (linear) equations.
9. Evaluation of the redundant reaction force and moment acting at point A (`XA, YA, CA`) starting from the system of equation imposed by the kinematic congruence equations (see point 8).

Castigliano's Theorem

CASE D: Roll bar - solution



10. Evaluation of the overall internal energy of the structure *U*, substituting the definition of `XA, YA, CA`. The U relation is function of the external load *P* and of the fictitious force *F* acting at B.
11. The displacement at the B point is evaluated through the Castigliano's theorem;
12. The fictitious nature of *F* may now be enforced to be null.

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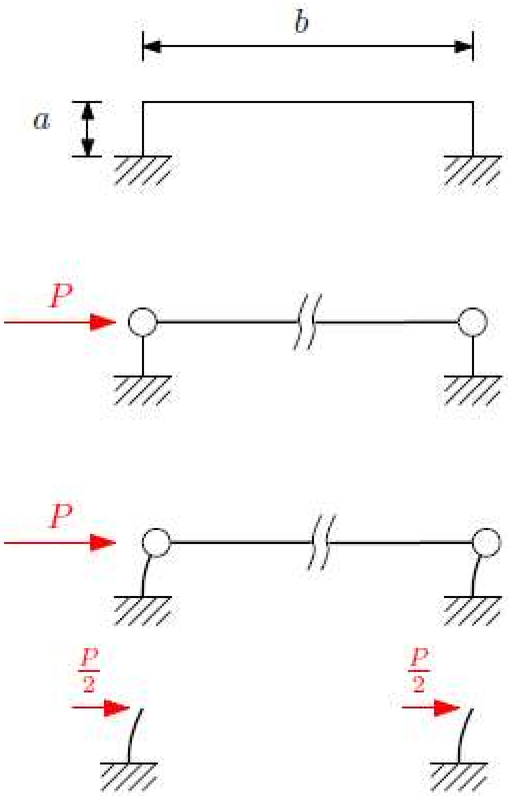
Castigliano’s Theorem

Considerations Case 1: b >>> a



The horizontal beam shows high bending deformation, however it is rigid at the normal force, therefore it might be approximated by a rod.

The vertical elastic beams work on parallel and the external loading P is equally subdivided.

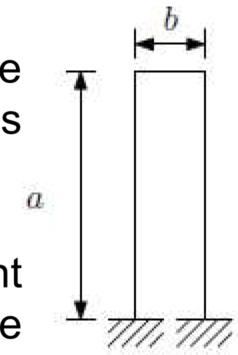
$$dC_{ref} = \frac{a^3}{3EJ}P;$$

$$K_{ref} = \frac{3EJ}{a^3}$$

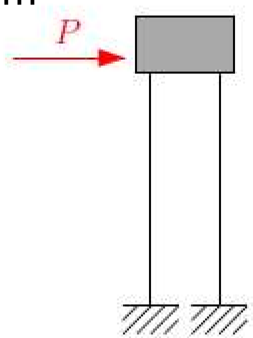
Castigliano's Theorem

Considerations Case 2: b<<<a

The horizontal beam to its limited length, is rigid on the flexural deformation, therefore might be considered as a rigid body.



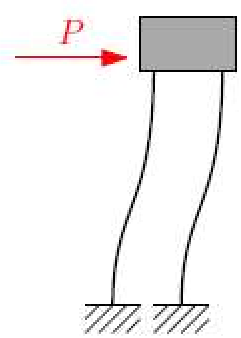
The vertical beams deforms similarly to a redundant beam structure (+1dof), fixed at one end and on the other end constraints with a double-double pendulum (*ddp*). The lateral external load at the latter extremity.

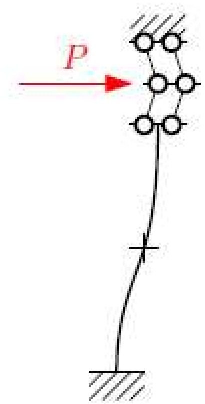


$$K_{cantilever} = \frac{3EJ}{a^3}$$

$$K_{fixed-ddp} = 4 * K_{cantilever}$$

Therefore, the stiffness of the rollbar structure at this configuration might be rationalized as two beam with ddp and fixed BCs.





$$K_{rollbar} = 2 * K_{fixed-ddp} = 2 * (4 * \ K_{cantilever}) \\ = 8 * K_{cantilever}$$

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Castigliano’s Theorem analysis of a rollbar (Case D) by Maxima

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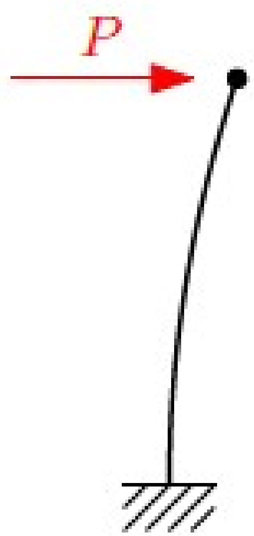
Castigliano's Theorem

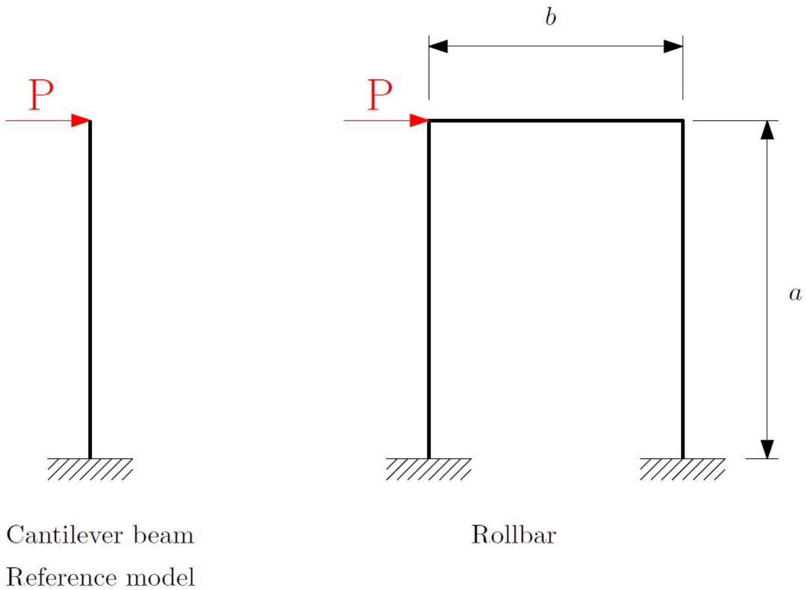
Considerations Case 3: b = 0



In this case, the two vertical beams coincide therefore this peculiar structure might be represented by a beam with stiffness doubles than the single cantilever beam.

Castigliano's Theorem

Considerations Case 3: b = 0



$$K = \frac{P}{dC}$$

$$K_{ratio} = \frac{K_{rollbar}}{K_{ref}} = \frac{dC_{ref}}{dC_{rollbar}} = \frac{dC_{ref}}{dC}$$

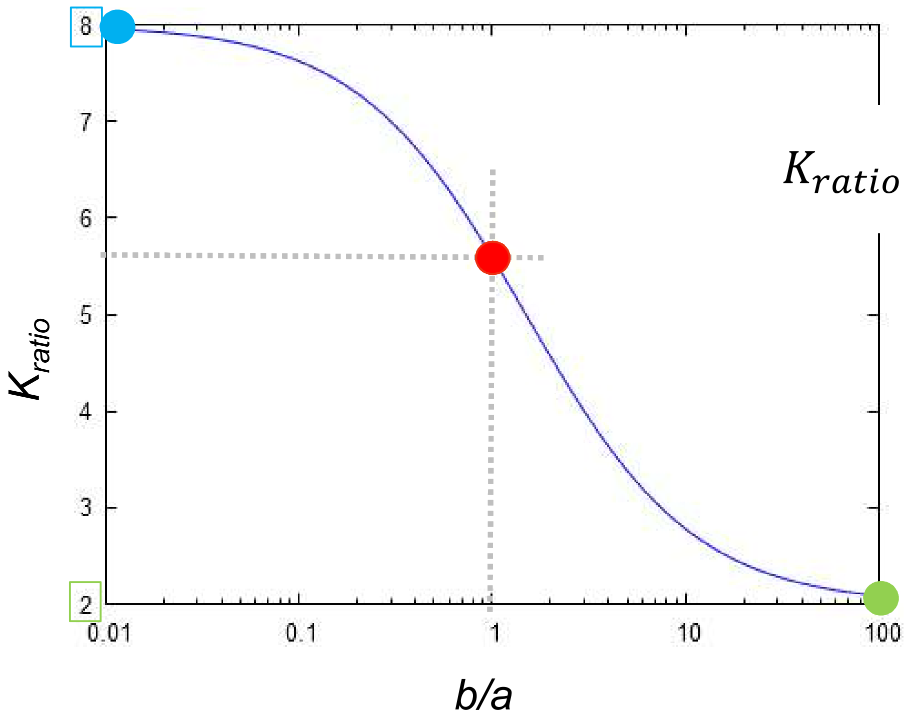
$$dC_{ref} = \frac{a^3}{3EJ}P;$$

$$K_{ref} = \frac{3EJ}{a^3}$$

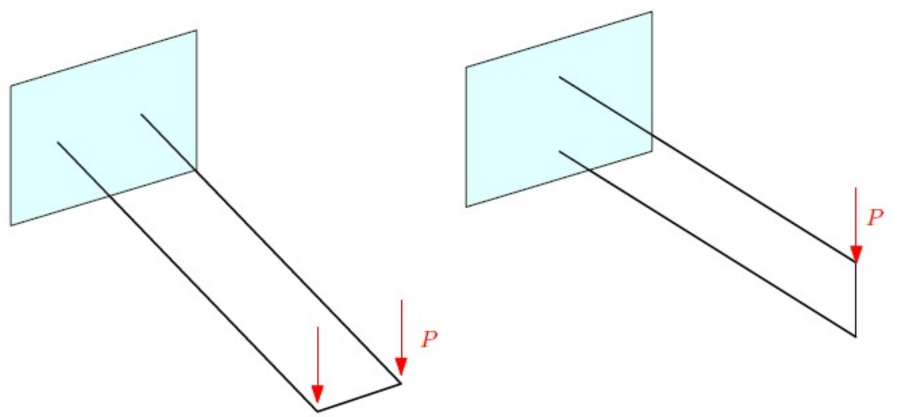
Where *dC* has been evaluated by the rollbar Maxima program.

Castigliano's Theorem

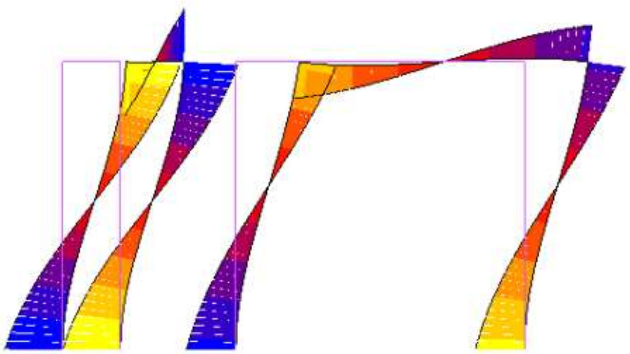
Considerations



$$K_{ratio} = \frac{K_{rollbar}}{K_{ref}} = \frac{dC_{ref}}{dC_{rollbar}} = \frac{dC_{ref}}{dC}$$

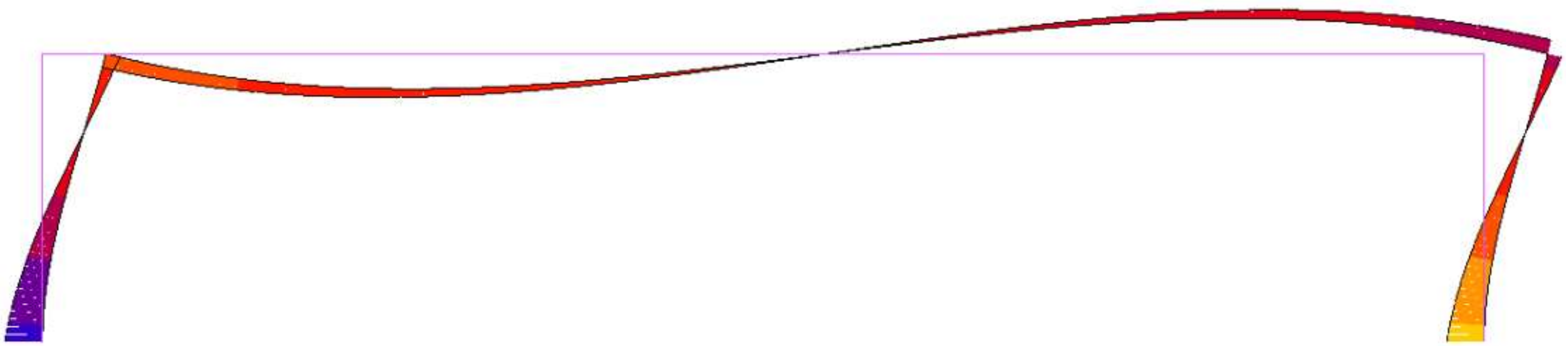


Castigliano's Theorem
Influence of the beam aspect ratio



The three frame structures geometry and their stiffness is reported below. Their stiffness is compared to the stiffness of the single upright member, taken as a reference (k_{ref}):

- $b/a = 1/5$: $k = 7.29 k_{\mathrm{ref}}$
- $b/a = 1/1$: $k = 5.60 k_{\mathrm{ref}}$
- $b/a = 5/1$: $k = 3.38 k_{\mathrm{ref}}$



Deformed and undeformed shapes are plotted, plus the bending moment. Beam cross section is uniform along the frame, and very slender.

Agenda

Castigliano's Theorem analysis of a rollbar (Case D) by Maxima

Centroid and Shear center

- Generic and peculiar cross-sections
- An automotive sill
- Further considerations

Symmetry BCs

Skew-symmetry BCs

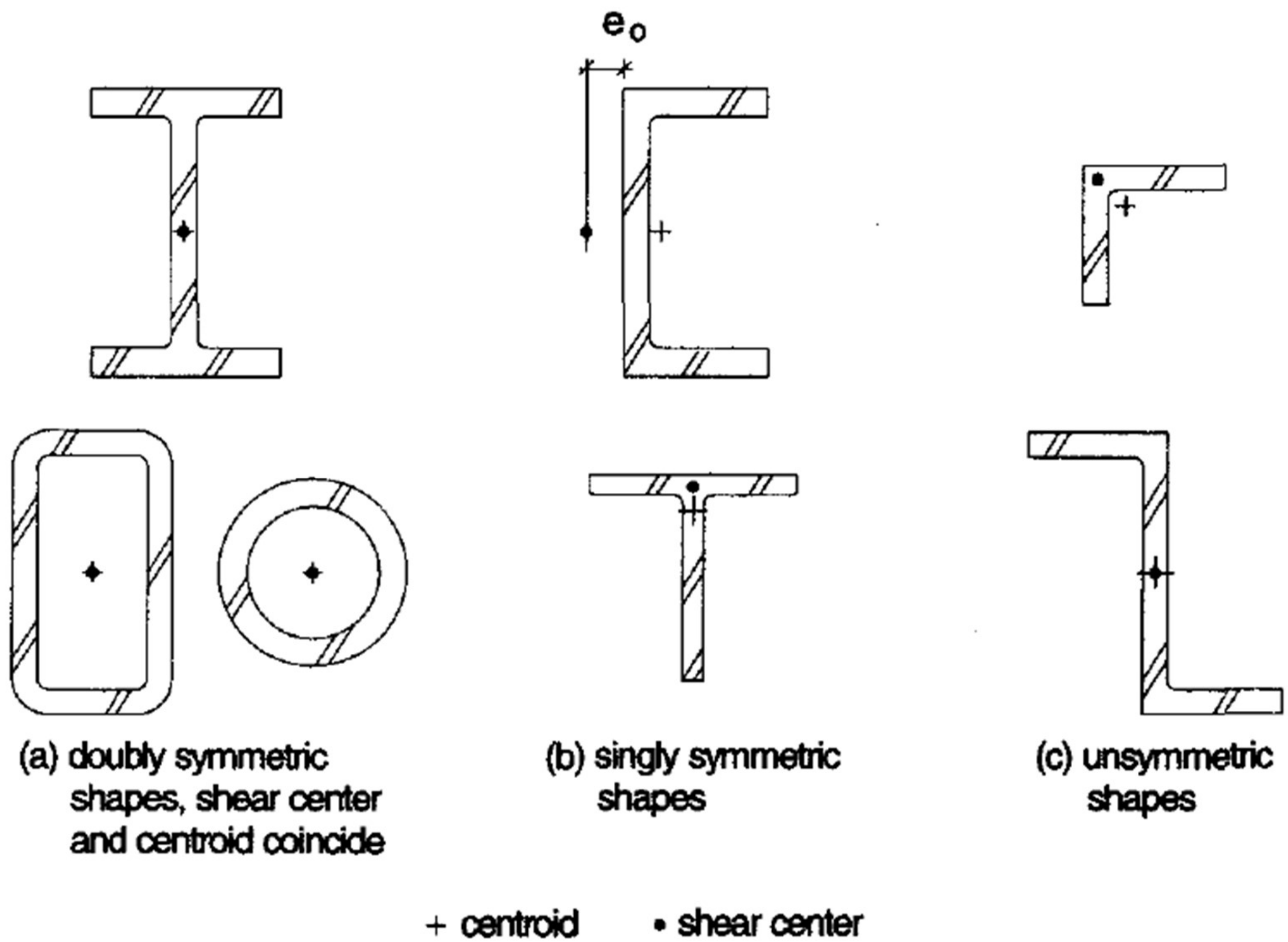
Symm and skew-symmetry BCs

Modelling technique

References

Centroid and Shear center

Generic and peculiar cross-sections



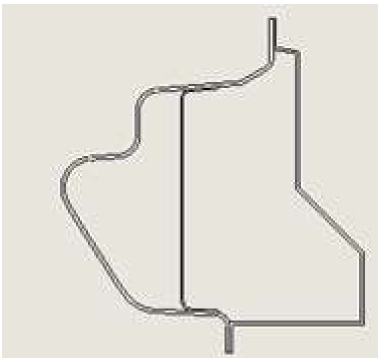
Centroid and Shear center

Generic and peculiar cross-sections

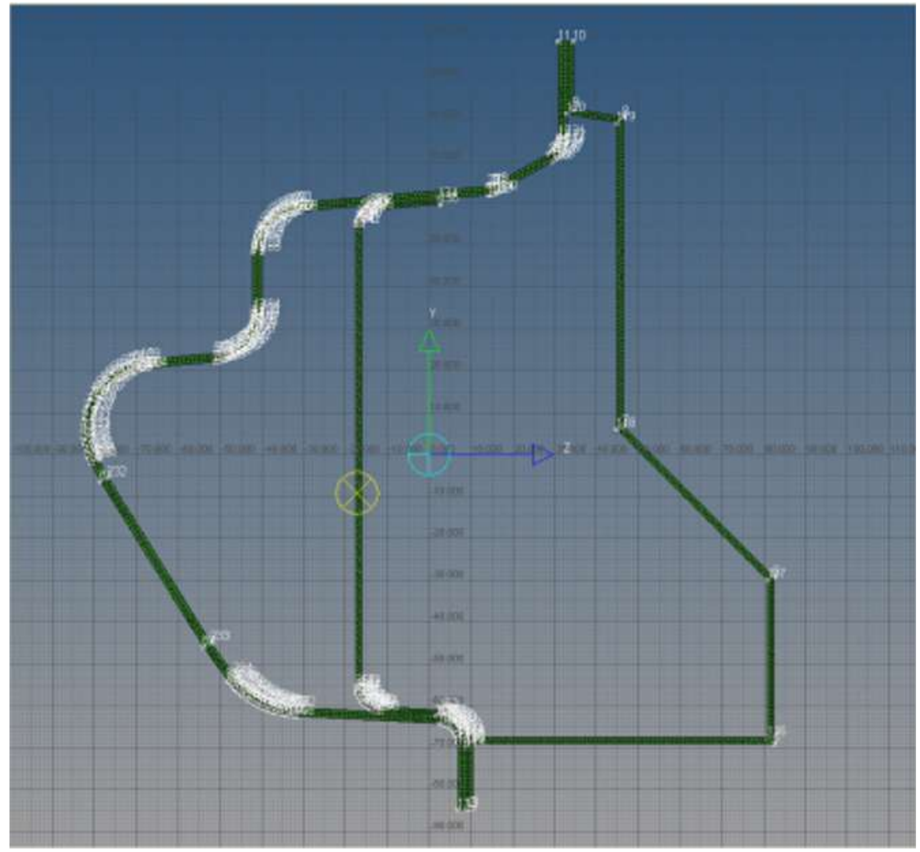
Cross sections with:

1) at least two planes of symmetry, the centroid and the shear center coincide;
2) only one plane of symmetry, the centroid and the shear center lies on the axis of symmetry, these points do **not** coincide;
3) a generic section, the evaluation of the shear center is not a trivial structural problem, therefore engineering books provide the shear center positioning for the more common cross-sections, or advanced calculations by either an algebraic manipulator, or by Finite Element modelling can be performed.

Centroid and shear center
An automotive sill



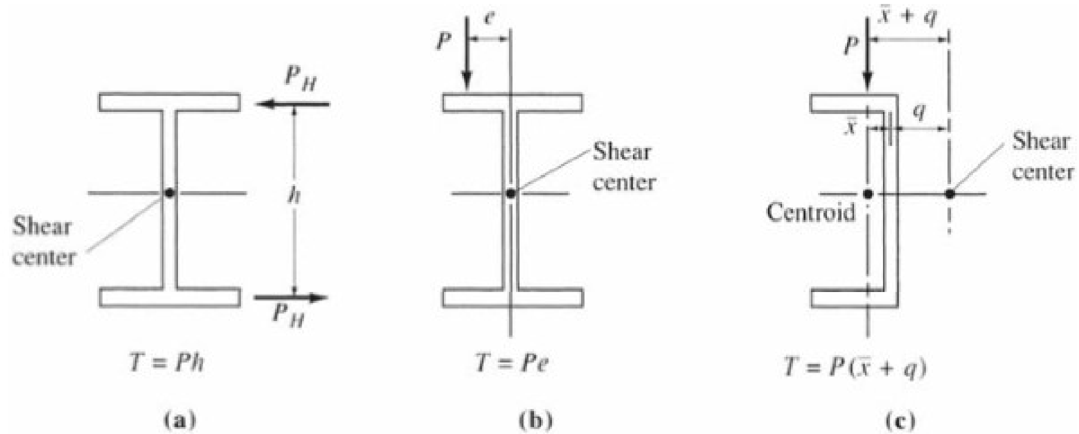
Summary	Door sill	
Area	1,25E+03	mm^2
Moments Of Inertia :		
Centroidal		
Iy	2,65E+06	mm^4
Iz	3,58E+06	mm^4
Iyz	-7,52E+04	mm^4
Principal		
Iv	2,64E+06	mm^4
Iw	3,59E+06	mm^4
Angle	-8,02E-02	rad
Polar	6,23E+06	mm^4
Radius of Gyration	4,60E+01	mm
Torsional Constant	3,12E+06	Nmm/(rad/mm)



⊗: Center of gravity
⊗: Shear center

Centroid and Shear center

Further considerations



If the shear force act at the shear center, the stresses are related on shear stress alone; therefore the this force does not induce a rotation of the beam along its axis. The beam deforms by bending alone.

For beam cross-section with a one axis of symmetry alone and a force acting outside the shear center, the beam deformation is a combination of a torsional and bending deformation.

Agenda

Castigliano's Theorem analysis of a rollbar (Case D) by Maxima

Centroid and Shear center

Symmetry BCs

- Why do you adopt symmetry?
- Loading
- Constraints

Skew-symmetry BCs

Symm and skew-symmetry BCs

Modelling technique

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Symmetry BCs

Why do you adopt symmetry?

In most cases, utilizing as many planes of symmetry as are allowed by the problem will result in shorter run times, more accurate boundary conditions, and more accurate solutions deriving from the previous two benefits.

Any 3D model can have a maximum of three orthogonal planes of symmetry in which the geometry, properties, and boundary condtions are equivalent across these planes.

An object has reflectional symmetry (line or mirror symmetry) if there is a line going through it which divides it into two pieces which are **mirror images** of each other.

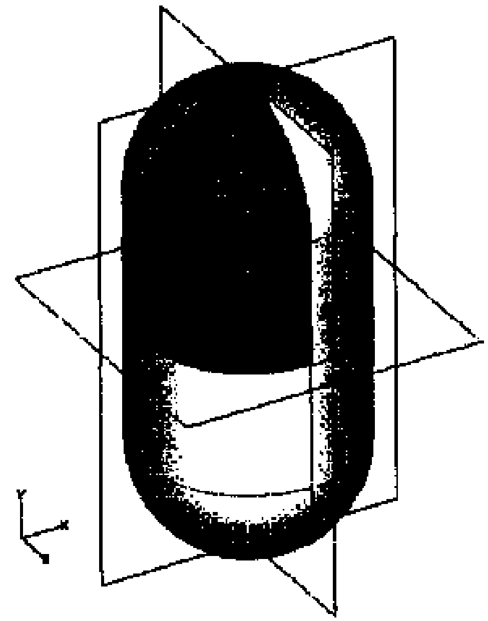


Fig. 4.15. Three symmetry planes for a domed pressure vessel with the section required for an internal pressure load.

Symmetry BCs

Why do you adopt symmetry?

Considering the three planes of symmetry, the vessel could be modelled with only one eighth of the structure.

Symmetry conditions require that the the geometry, and the boundary conditions are equal across one, two, or three planes.

→ Loading and constraints symmetry definition.

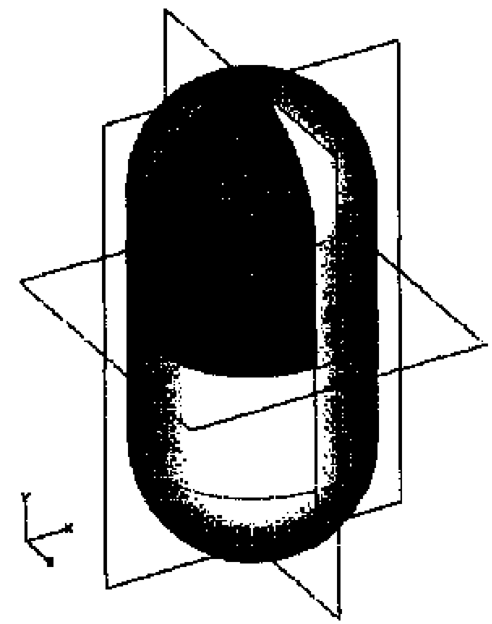


Fig. 4.15. Three symmetry planes for a domed pressure vessel with the section required for an internal pressure load.

Symmetry BCs

Loading

The loading applied to a symmetric model should be divided by the number of the symmetry planes used:

N. of planes	Magnitude of the symmetry loading condition
1	½ F
2	¼ F
p = F/A	???

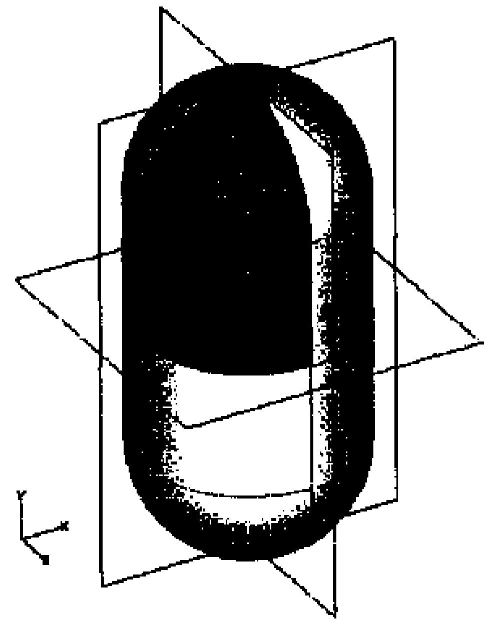


Fig. 4.15. Three symmetry planes for a domed pressure vessel with the section required for an internal pressure load.

A pressure load (*p*) will automatically halve or quarter themselves due to the available surface area considered in the symmetric model.

NOTE: a check of the total load and the magnitude of the loading to be applied for a symmetrical model could be thougth by the mirror images concept.

Symmetry BCs

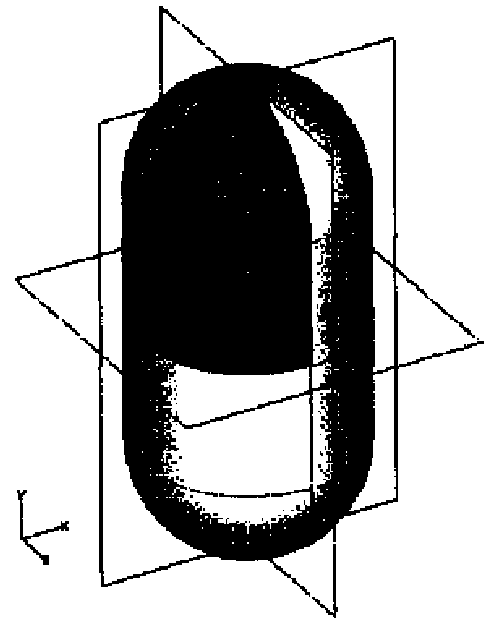
Constraints

The constraints on:

1) a **solid model** must prevent traslation throught the plane of symmetry on the entire cut face;

2) On beam and shell elements must also prevent rotation in the components parallel to the cut planes.

These constraints ensure tangency and continuity at the cut plane, just as the other half of the model would if it existed.



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Castigliano's Theorem analysis of a rollbar (Case D) by Maxima

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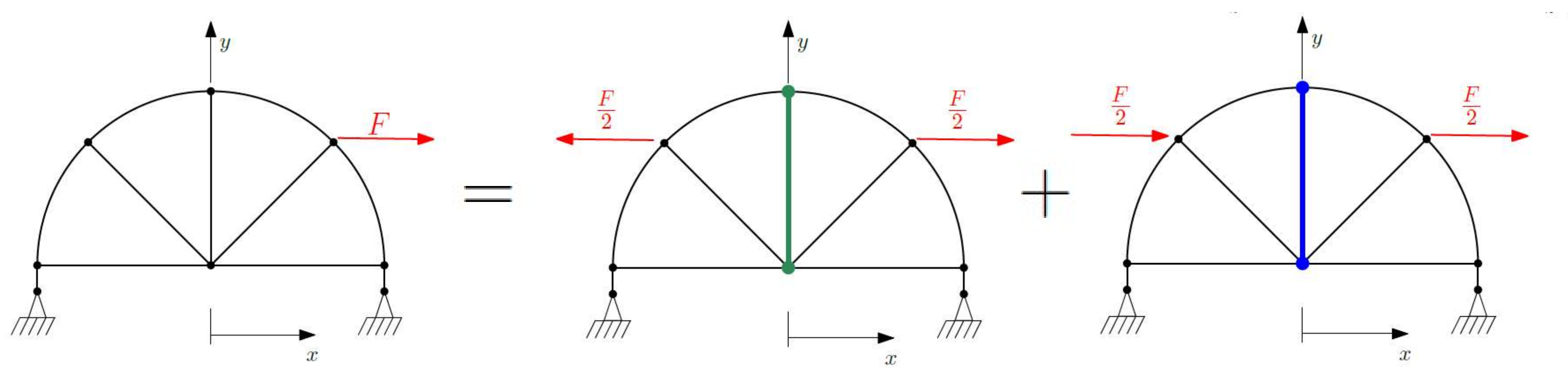
Skew-symmetry BCs

What are these BCs?

These constraints are not so intuitive as the previously described symmetry conditions. In case of skew-symmetry, a constraint equivalent to a *doweled* ***sphere* - *slotted cylinder*** joint, where the guide axis is orthogonal to the skew-symmetry plane, is applied at the points belonging to the intersection between the deformable body and the plane.

This technique can be used when the geometry conforms to planar symmetry however the loading does not.



Simmetry Conditions
$yz-$ symm plane

Skew-Simmetry Conditions
$yz-$ skew symm plane

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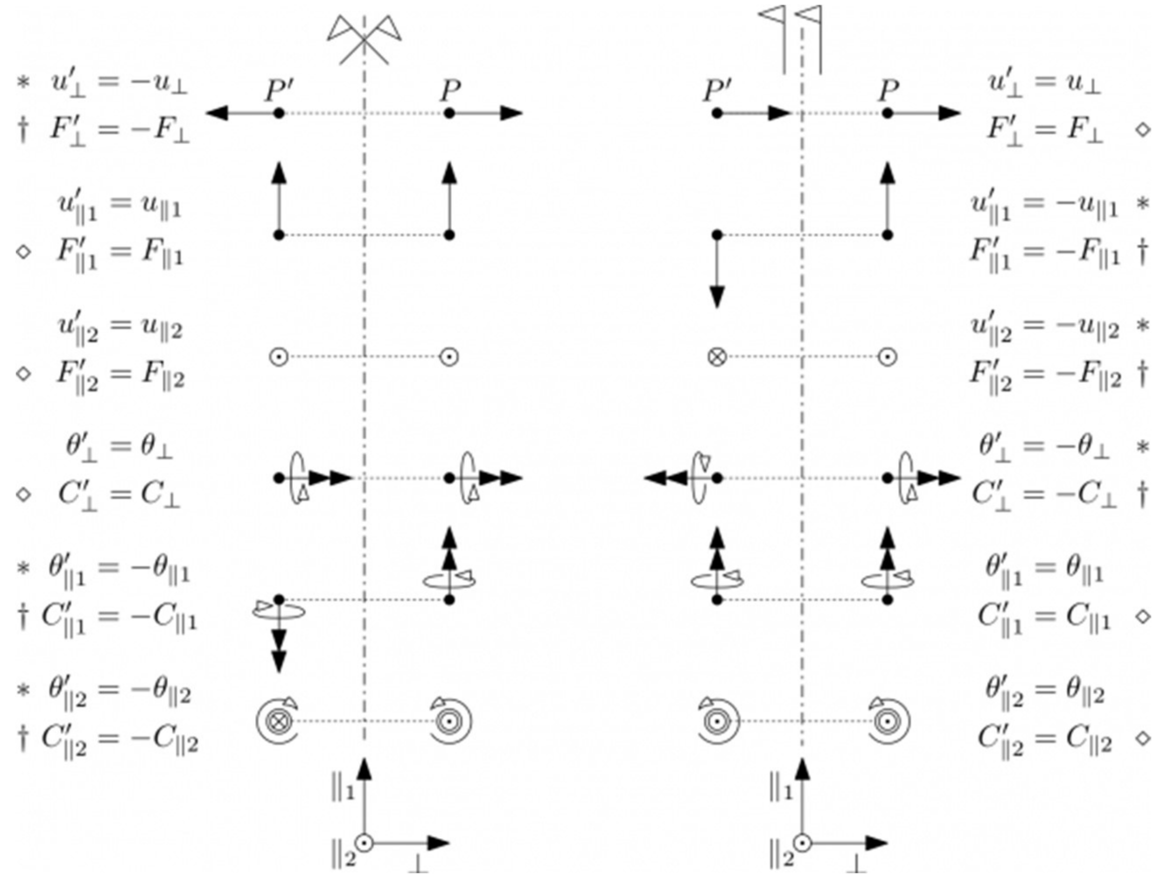
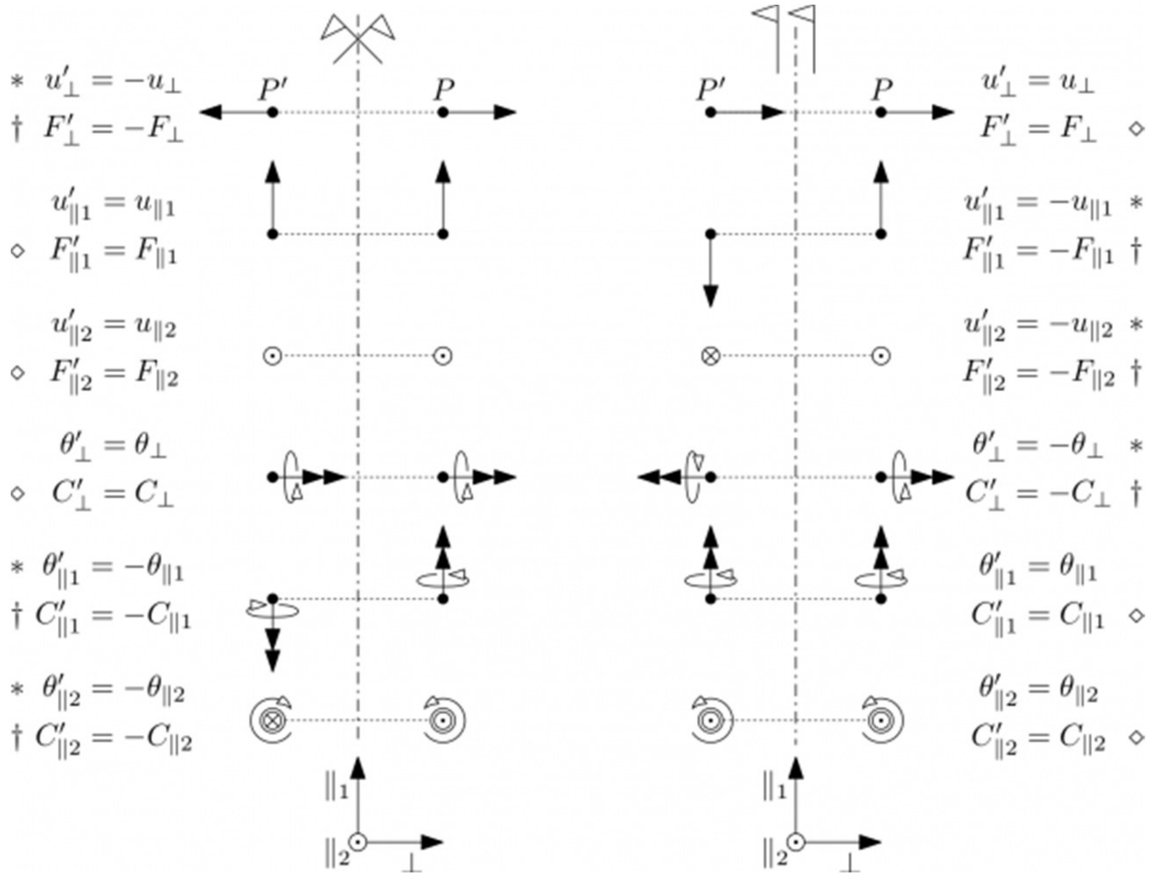
Symmetry and skew-symmetry BCs



The ∗ (generalized) displacement/rotation components may induce material discontinuity at points laying on the [skew-]symmetry plane, if nonzero.
They have to be **constrained to zero** value at those points, thus introducing [skew-] symmetry constraints. These constraints **act in place** of the portion of the structure that is omitted from our model, since the results for the whole structure may be derived from the modeled portion alone, due to [skew-]symmetry.

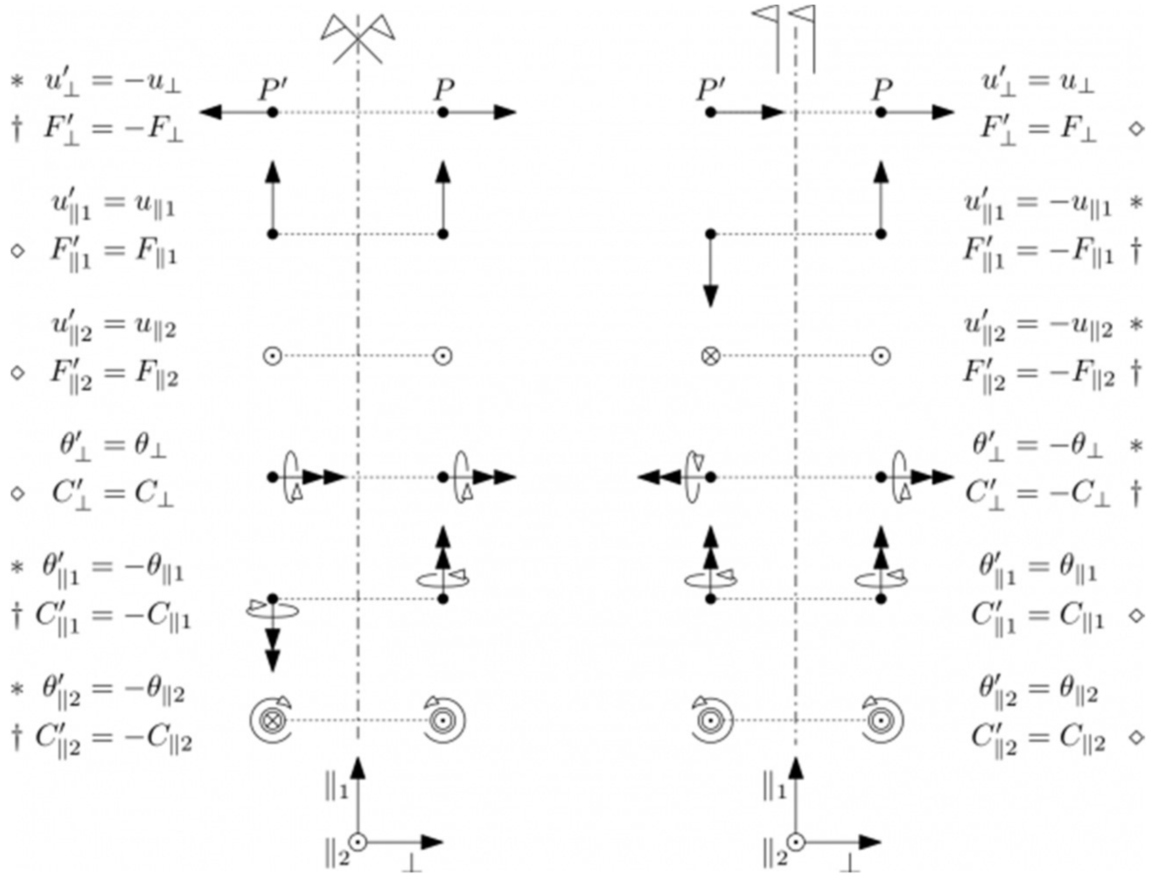
Symmetry and skew-symmetry BCs



The ⋄ internal action components are null at points pertaining to the [skew-]symmetry plane, since they would otherwise violate the action-reaction law.

The complementary † internal action components are generally nonzero at the [skew-]symmetry plate.

Symmetry and skew-symmetry BCs



The † external action components are not allowed at points along the [skew-]symmetry plane; instead, the complementary ⋄ generalized force components are allowed, if they are due to external actions.

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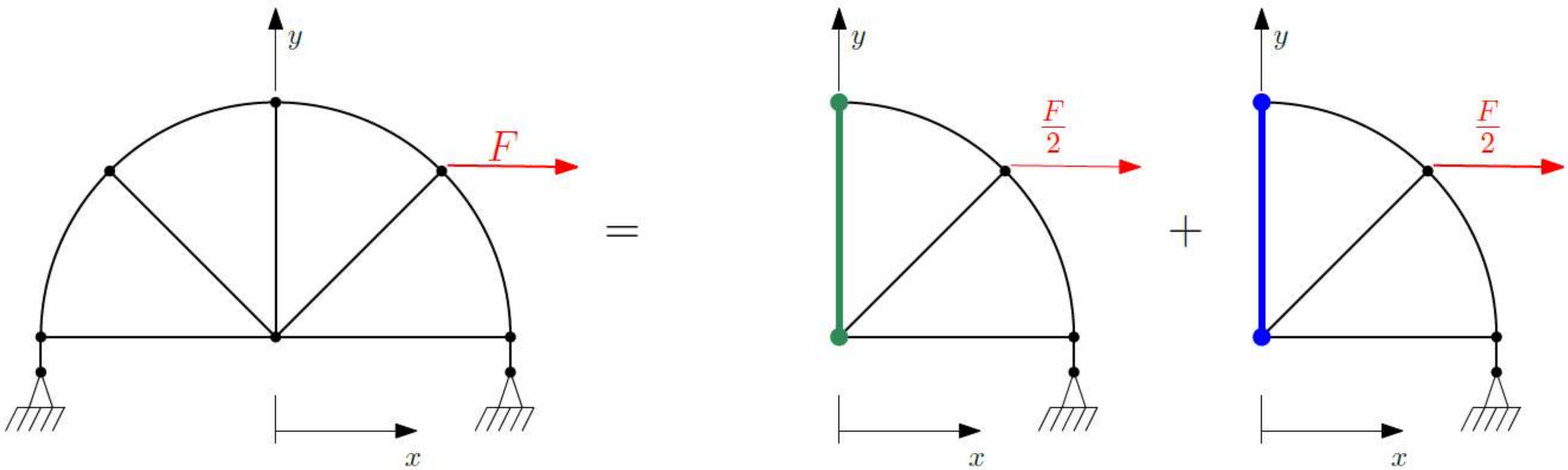
Modelling symm-skew symm decomposition

Modelling a plane frame problem as the sum of symm and skew-symm case



Simmetry Conditions

$u_x = 0$
$u_y \neq 0$
$u_z \neq 0$
$\theta_x \neq 0$
$\theta_y = 0$
$\theta_z = 0$

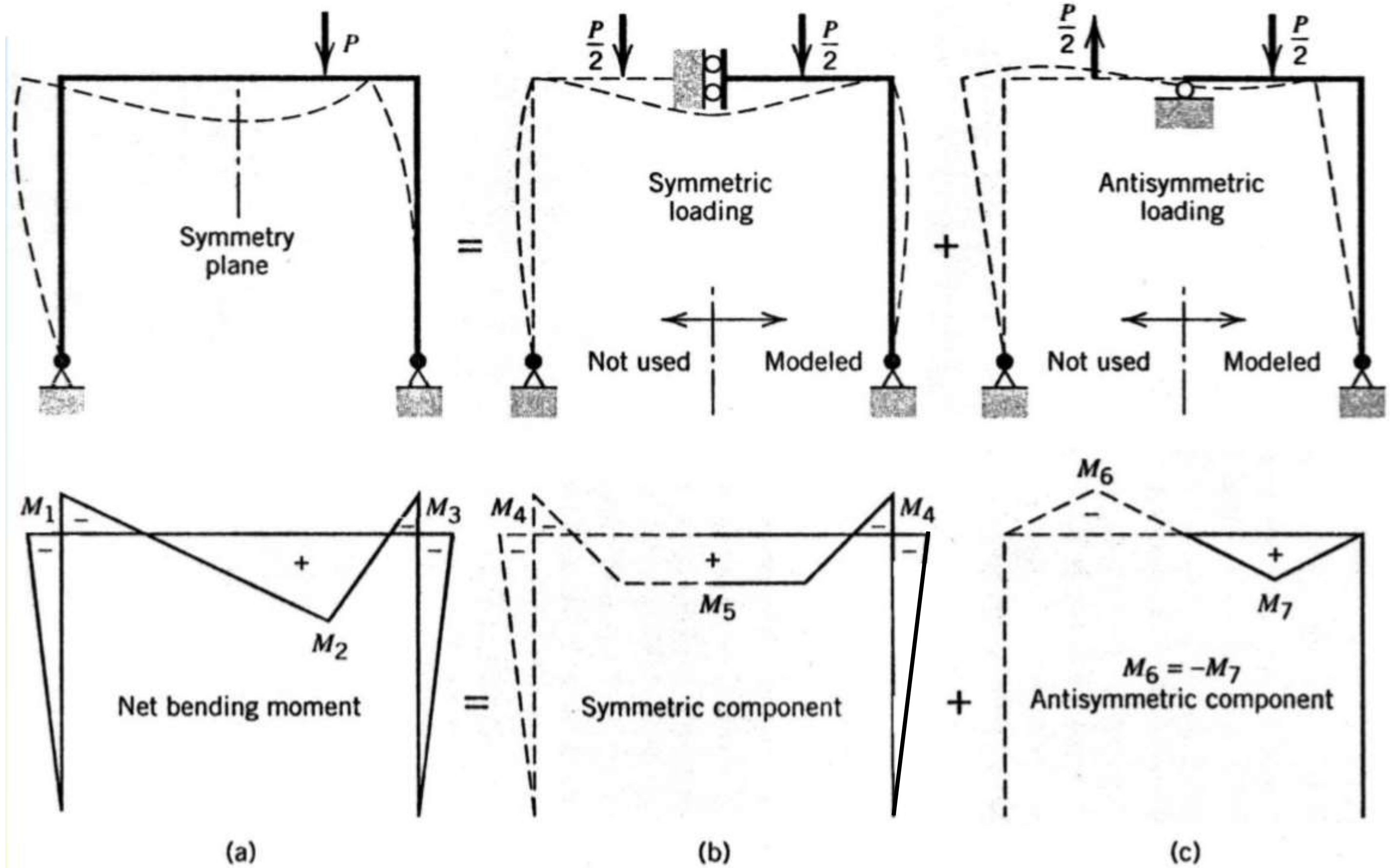
Skew-Simmetry Conditions

$u_x \neq 0$
$u_y = 0$
$u_z = 0$
$\theta_x = 0$
$\theta_y \neq 0$
$\theta_z \neq 0$

In the case of a symmetric structure, generally asymmetric applied loads may be decomposed in a symmetric part and in a skew-symmetric part; the problem may be solved by employing a half structure model for both the loadcases; the results may finally be superposed since the system is assumed linear.

Modelling symm-skew symm decomposition

Modelling a plane frame problem as the sum of symm and skew-symm case



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LAB Maxima files saved as:
`rollbar_def.wxmx`

On the Symmetry and skew symmetry modelling topic:

Adams, Vince, and Abraham Askenazi. *Building better products with finite element analysis*. Cengage Learning, 1999, pp. 115-135.

"Life is like riding a bicycle. To keep your balance, you must keep moving"

A. Einstein

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