0.1 Beam axis and cross section definition

[1] A necessary condition for identifying a portion of deformable body as a beam – and hence applying the associated framework – is that its centroidal curve is at least loosely recognizable.

Once such centroidal line has been roughly defined, locally perpendicular planes may be derived whose intersection with the body itself defines the local beam cross section. Then, the G center of gravity position may be computed for each of the local cross sections, thus defining a second, refined centroidal line. A potentially iterative definition for the beam centroidal axis[2] is hence obtained.

A rather arbitrary orientation may then be chosen for the centroidal curve.

A *local* cross-sectional reference system may be defined by aligning the normal z axis with the oriented centroidal curve, and by employing as the first in-section axis, namely x, the projection onto the cross-section plane of a given global $\hat{v}$ vector, that is assumed to be not parallel to the beam axis.

The second in-section axis y is then derived, in order to obtain a local $Gxyz$ right-handed coordinate system, whose unit vectors are $\hat{\imath}, \hat{\jmath}, \hat{k}$.

Such construction of the local reference system for the beam branch is consistent with most the Finite Element (FE) codes.

If a thin walled profile is considered in place of a solid cross section member – i.e., the section wall midplane is recognizable too (see paragraph XXX below), then a curvilinear coordinate s may be defined that spans the in-cross-section wall midplane. Such in-cross-section wall midplane consists in a possibly multi-branched curve, which is parametrically defined by a pair of $x(s), y(s)$ functions, with s spanning the conventional $[0, l]$ interval.

In the case material is homogeneous along the wall thickness, the local thickness value $t(s)$ is some relevance, along with a local through-wall-thickness coordinate $r \in [-t(s)/2, +t(s)/2]$.

[2] here, centroidal curve, centroidal line, centroidal axis, or simply beam axis are treated as synonyms.

Such s, r, in-section coordinates based on the profile wall may be employed in place of their cartesian x, y counterparts, if favourable.

0.2 Joints and angular points

Beam axis may be discontinuous at sudden body geometry changes; a rigid body connection is ideally assumed to restrict the relative motion of the proximal segments.

Such rigid joint modeling may be extended to more complex n-way joints; if the joint finite stiffness is to be taken into account, it has to be described through the entries of a rank $6(n-1)$ symmetric square matrix [3].

At joints and at the beam axis angular points the cylindrical bodies obtained by sweeping the cross sections along the centroidal curve branches do usually overlap, and in general they only loosely mimic the actual deformable body geometry.

The results obtained through the local application of the elementary beam theory are of a problematic nature; they may at most be employed to scale the triaxial local stress/strain fields[4] that are evaluated resorting to more complex modelings.

0.3 Cross-sectional resultants for the spatial beam

At any point along the axis the beam may be notionally split, thus obtaining two facing cross sections, whose interaction is limited to three components of interfacial stresses, namely the axial normal stress σ_{zz} and the two shear components τ_{yz}, τ_{zx}.

Three force resultant components may be defined by integration along the cross section area, namely the normal force, the y- and the

[3] i.e., joint stiffness is unfortunately not a scalar value.

[4] The peak stress values obtained through the elementary beam theory may be profitably employed as *nominal stresses* within the stress concentration effect framework.

x- oriented shear forces, respectively defined as

$$N = \int_A \sigma_{zz} dA$$
$$Q_y = \int_A \tau_{yz} dA$$
$$Q_x = \int_A \tau_{zx} dA$$

Three moment resultant components may be similarly defined, namely the x- and y- oriented bending moments, and the torsional moment. However, if the centroid is the preferred fulcrum for evaluating the bending moments, the below discussed C shear center is employed for evaluating the torsional moment; the two points might coincide, e.g. if the cross section is twice symmetric, but they are distinct in general. We hence define

$$M_x \equiv M_{(G,x)} = \int_A \sigma_{zz} y dA$$
$$M_y \equiv M_{(G,y)} = -\int_A \sigma_{zz} x dA$$
$$M_t \equiv M_{(C,z)} = \int_A \left[\tau_{yz}(x - x_C) - \tau_{zx}(y - y_C)\right] dA$$

The applied vector associated to the normal force component $(G, N\hat{k})$ is located at the section center of gravity , whereas the shear force $(C, Q_x\hat{\imath} + Q_y\hat{\jmath})$ is supposed to act at the shear center; such convention decouples the energy contribution of force and moment components for the straight beam.

Common alternative names for such resultants are *component of internal action*, *(beam) generalized stress components* etc.; they may also be interpreted as the reactions of an internal clamp constraint that joins the upstream and downstream portions of the structure, notionally severed at the cross section under scrutiny.

Most of the sign rules for the resultant force and moment components introduced for the plane problem lose their significance in the spatial realm.

The following convention is proposed for the few cases in which a sign characterization for the stress resultant components is required,

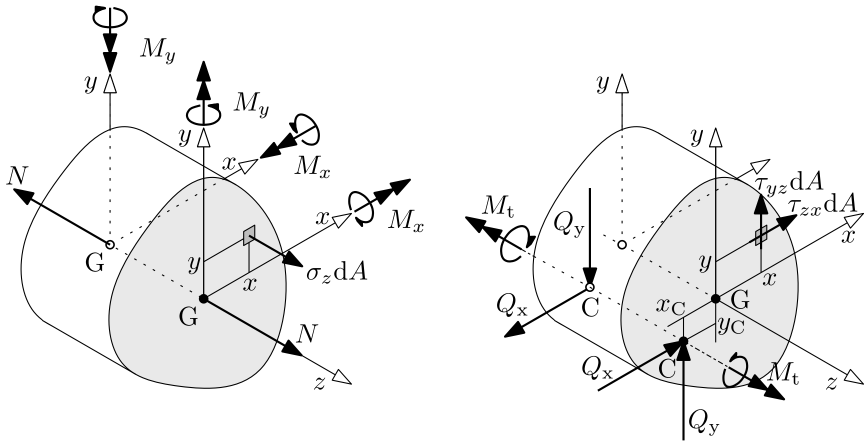

Figure 1: Stress resultants for the beam segment and the associated sign convention; for the sake of readability, symmetric and skew-symmetric components are split apart in Figure. Please remind that – even if visually applied at notable locations – the moment components have no definite application point within the cross section.

which originates from the definition of the local reference system, which in turn derives from the oriented nature of the beam branch, and from the $\hat{v}$ orientation vector, as discussed above; such rule is widely employed by FE codes.

Let's consider to the beam segment of Fig. 1 (a) and (b): positive resultant components adopt the direction of the associated local axis at the beam segment end that shows an outward-oriented local z axis; at beam segment ends characterized by an inward-oriented local z axis, the same positive stress resultant components are counter-oriented to the respective local axes.

According to such a rule, axial load is positive if tractive, and the torsional moment is positive if deflects into a right helix a line traced parallel to the axis on the undeformed profile. No intuitive formulations are however available for the bending moment and shear components.

Cross section resultants may be obtained, based on equilibrium for a statically determinate structure. The ordinary procedure consists in

- notionally splitting the structure at the cross section whose resultants are under scrutiny;

- isolating a portion of the structure that ends at the cut, whose locally applied loads are all known; the structure has to be preliminarily solved for the all the constraint reactions that act on the isolated portion;

- setting the equilibrium equations for the isolated substructure, according to which the cross-sectional resultants are in equilibrium with all the loads locally applied to the isolated portion.

0.4 A worked example

The present paragraph is devoted to the evaluation of the stress resultants along the BD beam segment[5] of the simple structure of Figure 2c, which mimics from within the spatial beam framework boundaries the deformable body of Figure 2a.

The assumed distribution for the shear stress components τ_{zx} and τ_{yz} along the C-section thin wall, which is derived from a generalized application of the Jourawsky shear theory, locates its resultants in a shear center C which is external to the cross section convex envelope, as shown in Fig. 2b.

The shear center locus is represented in Fig. 2c as a dotted line, wherever distinct from the centroidal line.

The l distance from the B corner parametrically pinpoints a section along the BD segment, in correspondence of which the stress resultant components are evaluated.

The structure is then notionally partitioned in two substructures, and the portion spanning from the section under scrutiny to the free end is elected for further equilibrium analysis. Equilibrium equations for the other portion would involve the preliminary evaluation of the six constraint reaction components at D, based on global equilibrium.

Figure 2d collects the loads applied to such isolated substructure, including the six components of internal action at the section under scrutiny; the following equilibrium equations are set:

- translational equilibrium along the local x axis, namely

$$\mathrm{tx}: \; +F + Q_{\mathrm{x}} = 0;$$

[5] The more straightforward treatise of the AB segment is left to the reader; results will be here reported for discussion.

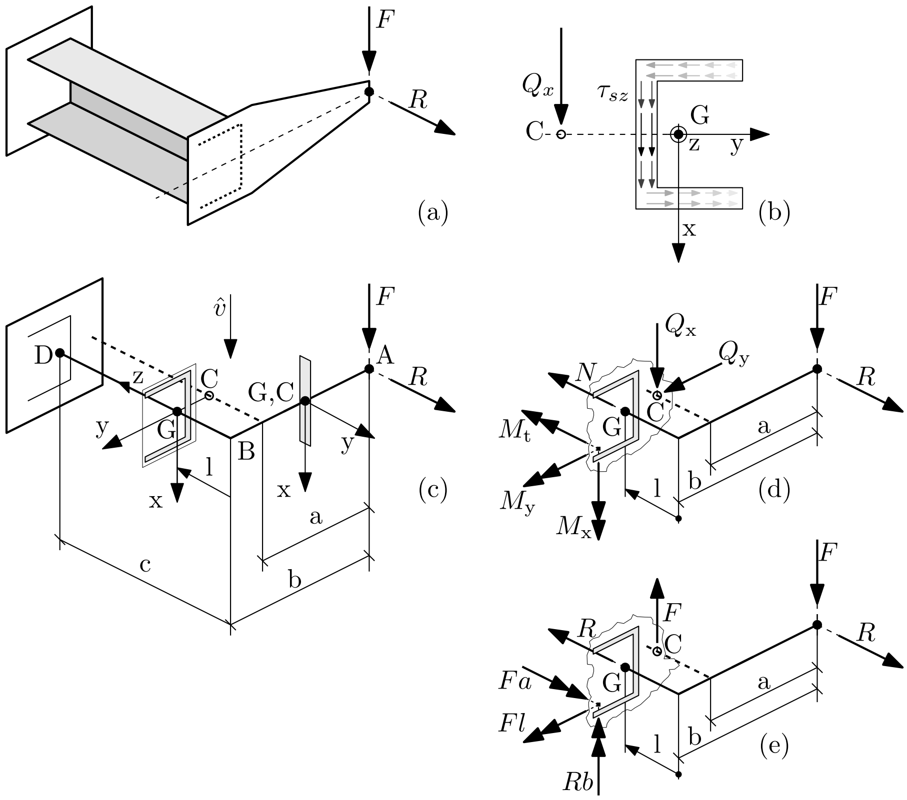

Figure 2: A planar beam structure, loaded both in-plane and out of plane. Please note that the plane the structure lies on is a symmetry plane for the material and for the constraints; the applied load may hence be decomposed into symmetric and skew-symmetric parts, leading to two uncoupled problems. A general spatial structure may be derived e.g. by turning the C-profile 90° on its axis.

- translational equilibrium along the local y axis, namely

$$\mathrm{ty}: \ +Q_{\mathrm{y}} = 0;$$

- translational equilibrium along the local z axis, namely

$$\mathrm{tz}: \ -R + N = 0;$$

- rotational equilibrium with respect to the centroidal, x-aligned axis, namely

$$\mathrm{rGx}: \ +Rb + M_{\mathrm{x}} = 0;$$

.

- rotational equilibrium with respect to the centroidal, y-aligned axis, namely

$$\mathrm{rGy}: \ -Fl + M_{\mathrm{y}} = 0;$$

.

- rotational equilibrium with respect to z-aligned axis passing through the shear center, namely

$$\mathrm{rCz}: \ +Fa + M_{\mathrm{t}} = 0;$$

from which the stress resultants may be trivially obtained.

The meditated choice for the rotational equilibrium axis makes the arm of the possibly unknown axial and shear forces vanish, thus decoupling the equations.

Also, it is suggested to analyze the contributions to the rotational equilibrium with respect to a given axis by resorting to a projected view of the isolated substructure in which such axis is aligned with the line of sight[6], see Figure 3; the information lost in the projection are in fact of null relevance for the rotational equilibrium under scrutiny.

Figure 2e depicts the equilibrium state of the isolated substructure, and the visual comparison with its 2d counterpart offers an overview for the components of internal action.

A few final remarks follow.

[6]i.e. a view in which such axis is exiting (or entering) the plane of view

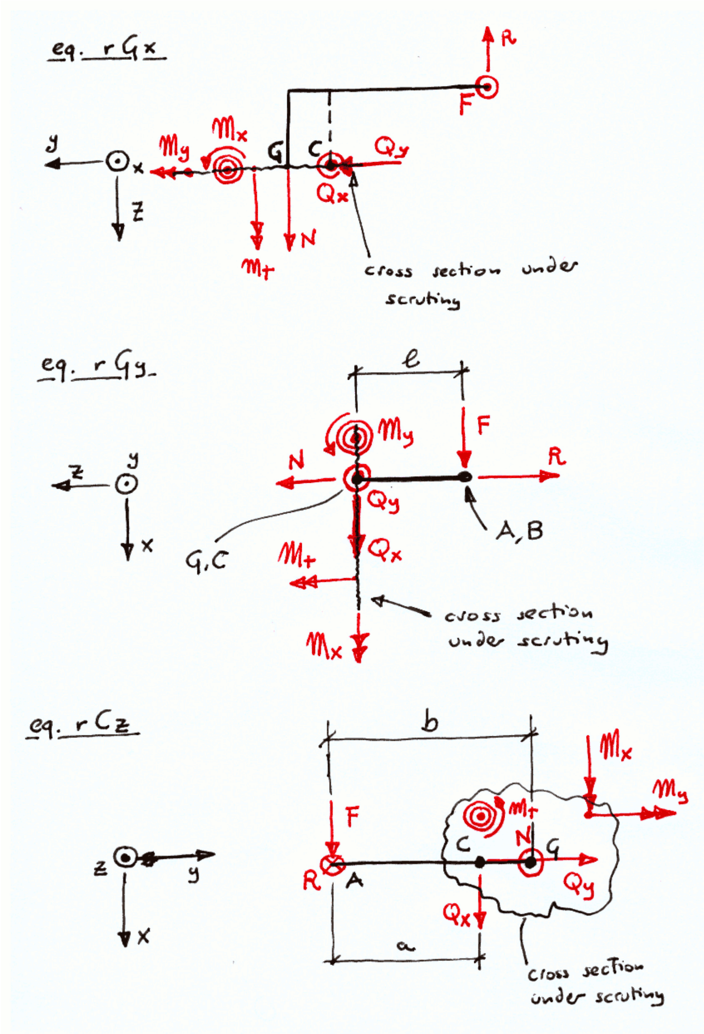

Figure 3: Projected views useful for discussing the isolated substructure rotational equilibrium. TODO.

The stress resultants at each section depend on the location of its center of gravity and shear center, and on the orientation of the local reference system; any variation of the cross section design which preserves the named elements does not require a reevaluation of the stress resultants.

Even if the described procedure is of general application within the spatial beam realm, the simple structure discussed exhibits elastic domain symmetry with respect to the plane the two centroidal segments lie on, a non-general property this, which is also respected by the specific constraints.

Such a peculiarity, along with the assumed linearity of the structure response, allows for the decomposition of the problem into a symmetric part, and into a skew-symmetric part. The symmetric portion of the applied load is embodied by the R force, whereas the skew-symmetric load portion is embodied by F.

Abetted by the fortunate orientation of the local axes[7] the three $N, Q_{\mathrm{y}}, M_{\mathrm{x}}$ in-plane resultants are produced by R alone, wherease the three $Q_{\mathrm{x}}, M_{\mathrm{y}}, M_{\mathrm{t}}$ out-of-plane resultants are induced by F alone. Inplane (out-of-plane) resultants are in fact symmetric (skew-symmetric) with respect to the plane the beam branches lies on, and the two symmetric and skew-symmetric parts of the problem are uncoupled.

Such property is useful in analyzing plane structures subject to mixed in-plane and out-of-plane loads, as the one under scrutiny.

It is finally noted that a general spatial structure may be derived from the proposed one e.g. by turning the C-profile 90° on its centroidal axis, and thus losing the elastic body symmetry.

[7] one parallel and one orthogonal to the symmetry plane

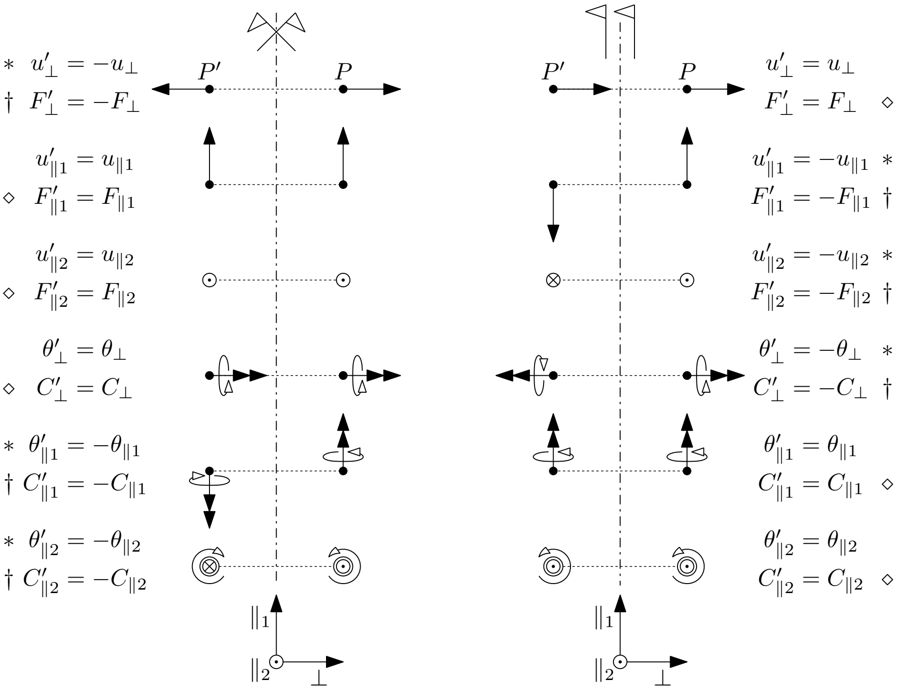

Figure 4: An overview of symmetrical and skew-symmetrical (generalized) loading and displacements.

0.5 Symmetry and skew-symmetry conditions

Symmetric and skew-symmetric loading conditions are mostly relevant for linearly-behaving systems; a nonlinear system may develop an asymmetric response to symmetric loading (e.g. column buckling).

Figure 4 collects symmetrical and skew-symmetrical pairs of vectors and moment vectors (moments); those (generalized) vectors are applied at symmetric points in space with respect to the reference plane. Vectors which are either normal or parallel to the plane are considered, that may embody the same named components of a generally oriented vector.

It may be observed that the symmetric/skew-symmetric condition for otherwise analogous pairs swaps in moving from vectors to moment vectors, and from the orthogonal to the parallel orientation.

The pair members may be moved towards the reference plane up to a vanishing distance ϵ; for null ϵ both the point and the image lie on the plane, and they coincide. In the case different (in particular, opposite and nonzero) vectors are associated to the two coincident pair members, the physical field that such vectors are assumed to represent (displacements, applied forces, etc.) is not single-valued at the reference plane; such condition deserves an attentive rationalization.

Vector and moment pairs in Figure 4 may embody, depending on the context, displaments (denoted as u), rotations (θ), forces (F) and moments (M); the latters may be both related to internal and external actions; in the following, the feasibility of nonzero magnitude pairs is discussed as the members approach the reference plane ($\epsilon \to 0$).

The (generalized) displacement components decorated with the $*$ marker may induce material discontinuity at points laying on the [skew-]symmetry plane, if nonzero. Except for specific cases in which the discontinuity is expected – e.g. or notionally infinitesimal openings at the symmetry plane – they have to be constrained to zero at those points, thus introducing the so-called [skew-]symmetry constraints.

When an halved portion of the structure is modeled in place of the whole, since the response is expected to be [skew-]symmetric, these constraints act in place of the portion of the structure that is omitted from our model, and their reactions may be interpreted as internal action components at the coupling interface between the two halves.

In case of symmetry, a constraint equivalent to a planar joint is to be applied at points laying on the symmetry plane for ensuring displacement/rotation continuity between the modeled portion of the structure, and its image. In case of skew-symmetry, a constraint equivalent to a *doweled sphere - slotted cylinder* joint (see Figure 0.5), where the guide axis is orthogonal to the skew-symmetry plane, is applied at the points belonging to the intersection between the deformable body and the plane.

The $\diamond$ internal action components are null at points pertaining to the [skew-]symmetry plane, since they would otherwise violate the action-reaction law. The complementary † internal action components are generally nonzero at the [skew-]symmetry plane.

The † external action components are not allowed at points along the [skew-]symmetry plane; instead, the complementary $\diamond$ generalized force components are allowed, if they are due to locally applied external actions.

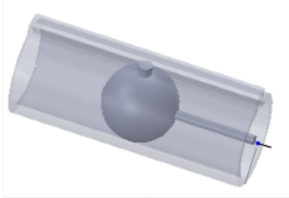

Figure 5: The doweled sphere - slotted cylinder joint, which is associated to the skew-symmetry constraint. In this particular application, the cylindrical guide may be considered as grounded.

In the case of a symmetric structure, generally asymmetric applied loads and imposed deflections may be decomposed in a symmetric part and in a skew-symmetric part; the problem may be solved by employing a half structure model for both the loadcases; the results may finally be superposed since the system is assumed linear.

0.6 Periodicity conditions

TODO, if required.