0.1 Castigliano's second theorem and its applications

Castigliano's second theorem may be employed for calculating deflections and rotations, and it states:

> Once the strain energy of a linear elastic structure is expressed as a function of a set of generalized loads[1] Q_i , the partial derivative of the strain energy with respect to each generalized load supplies the generalized displacement[2] q_i on which such a load performs work.

In equation form,

$$q_i = \frac{\partial U}{\partial Q_i}$$

where U is the strain energy.

In case of elastically nonlinear structures, the second Castigliano theorem may still be employed[3], provided that the complementary elastic strain energy U^* is used in place of the strain energy U, see Fig. 1. The two energy terms coincide in linearly behaving structures.

0.2 Internal energy for the spatial straight beam

The lineic[4] elastic strain energy density for the spatial rectilinear beam may be expressed as a quadratic function of its cross section resultants, thus leading to the general form

$$\frac{dU}{dl} = \frac{1}{2} \begin{pmatrix} N \\ M_\mathrm{x} \\ M_\mathrm{y} \\ Q_\mathrm{x} \\ Q_\mathrm{y} \\ M_\mathrm{t} \end{pmatrix}^\top \begin{pmatrix} a_{1,1} & g_{1,2} & g_{1,3} & i_{1,4} & i_{1,5} & i_{1,6} \\ 0 & b_{2,2} & e_{2,3} & i_{2,4} & i_{2,5} & i_{2,6} \\ 0 & 0 & b_{3,3} & i_{3,4} & i_{3,5} & i_{3,6} \\ 0 & 0 & 0 & c_{4,4} & f_{4,5} & h_{4,6} \\ 0 & 0 & 0 & 0 & c_{5,5} & h_{5,6} \\ 0 & 0 & 0 & 0 & 0 & d_{6,6} \end{pmatrix} \begin{pmatrix} N \\ M_\mathrm{x} \\ M_\mathrm{y} \\ Q_\mathrm{x} \\ Q_\mathrm{y} \\ M_\mathrm{t} \end{pmatrix}, \qquad (1)$$

[1] namely, forces or moments, but also a pressure load etc.

[2] namely displacements and rotations, or, in the case of a pressure load, the volume spanned with deformation by the pressurized surface.

[3] this nonlinear extension of the Castigliano theorem is however referred to in literature as the Crotti-Engesser theorem.

[4] i.e. per unit length; the far more customary *linear* adjective is so overloaded of meanings that I rather prefer such an exotic alternative.

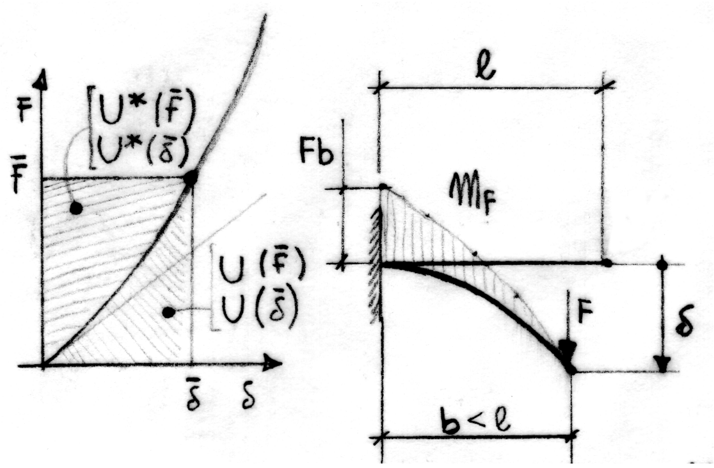

Figure 1: An elastic structure subject to large rotations, which shows a nonlinear stiffening behaviour; the bending moment diagram is evaluated based on the beam portion equilibrium in its *deformed* configuration. The complementary elastic strain energy U^* is plotted for a given applied load $\bar{f}$ or assumed displacement $\bar{\delta}$, alongside the elastic strain energy U.

where the coefficient matrix may be formally replaced by its symmetric part.

Most of the 21 independent matrix coefficients are zero if some properties hold for the beam cross section and material; in particular:

- the i coefficients are null if the material is symmetric with respect to the cross-sectional plane, i.e. if the material is *monoclinic* with respect to such a plane. An orthotropic material falls within this category if one of the principal directions is aligned with the beam axis. An isotropic material always falls within this category. Local scale material homogeneization may be considered for composite materials which are pointwisely not compliant, but compliant in average (e.g. through-thickness balanced laminates);

- the g and the h coefficients are also zeroed if, as it ordinarily happens, the poles employed in evaluating the bending moments and the torsional moment coincide with the centroid and with the shear center, respectively. Moreover, the coordinate of those two points, if not otherwise obtained, may be derived by imposing zero g and h coefficients.

- the e coefficient is zero if the local x, y axes are aligned with the principal directions of inertia of the cross section;

- the f coefficient is zero for a dedicated orientation choice for the shear force components, which *may or may not*[5] coincide with the principal directions of inertia; those directions coincide to the symmetry axes in the case of a twice symmetric cross section.

In the case of a homogeneus, isotropic material, the residual nonzero coefficients are defined as follows.

$$
\begin{aligned}
a_{1,1} =& \frac{1}{EA} & \{b_{2,2}, b_{3,3}, e_{2,3}\} =& \frac{\{J_{\mathrm{yy}}, J_{\mathrm{xx}}, 2J_{\mathrm{xy}}\}}{E\left(J_{\mathrm{xx}}J_{\mathrm{yy}} - J_{\mathrm{xy}}^2\right)} \\
d_{6,6} =& \frac{1}{GK_t} & \{c_{4,4}, c_{5,5}, f_{4,5}\} =& \frac{\{\chi_{\mathrm{x}}, \chi_{\mathrm{y}}, \chi_{\mathrm{xy}}\},}{GA}
\end{aligned}
$$

where

[5] TODO, please check for a proof, or a counter-example.

- A, J_{yy}, J_{xx} and J_{xy} are the section area and moments of inertia, respectively;
- K_{t} is the section torsional stiffness (**not** generally equivalent to its polar moment of inertia);
- E and G are the material Young Modulus and Shear Modulus, respectively.

The shear energy normalized coefficients χ_{y},χ_{x},χ_{xy} are specific to the cross section geometry, and may be collected from the expression of the shear strain energy due to the concurrent action of the $Q_{\mathrm{x}}, Q_{\mathrm{y}}$ shear forces.

In the case the strain energy contribution of any of the stress resultants is to be neglected, the associated matrix coefficients may be set to zero; such manipulation makes the beam *rigid* with respect to the stress resultant whose contribution to the strain energy is nullified.