7 Vlasov torsion theory

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Restrained Warping

The typical torsion stresses according to De Saint Venant only occur if warping can take place freely (Fig. 1). In engineering practice this rarely is the case. Warping can be restrained at supports, for example, a steel I-beam welded on a thick plate (Fig. 1, 2). Warping is also restrained at the position of an imposed torsion moment loading. The reason is that the loading gives a jump in the internal torsion moment M_t, which would give a jump in the warping if it could occur freely. Clearly, the sections left and right of the loading are attached, so warping there is restrained. Warping is also restrained where the cross-section changes (non-prismatic beams). Therefore, deviations from the torsion theory occur. If warping is completely prevented the local torsion stiffness GI_t is equal to GI_p.

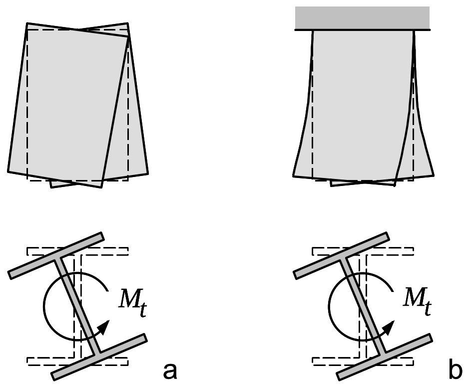


Figure 1. Free warping (a) and restrained warping (b) of an I-section loaded in torsion.

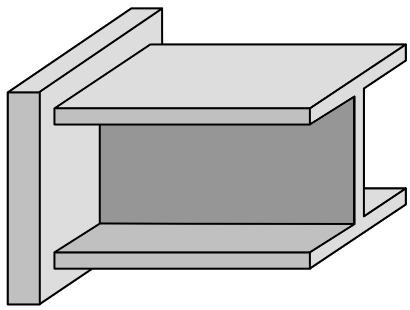


Figure 2. Prevented warping of a I-beam end

Another restriction of the Saint Venant theory is that a distributed moment m_x cannot occur (Fig. 3).

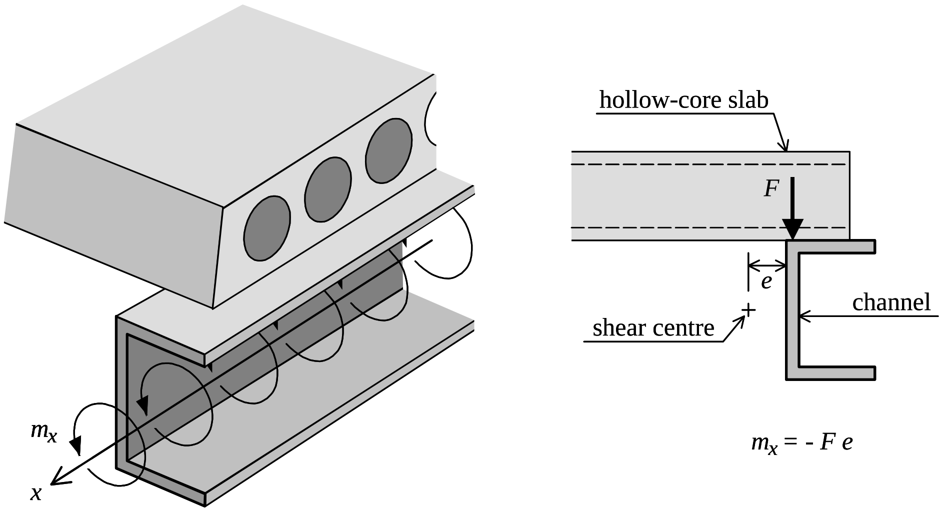


Figure 3. Distributed torsion moment loading on a channel due to hollow-core slabs

When warping is prevented locally axial stresses are introduced. These stresses diminish with the distance from the restrained but are noticeable over a considerable length of the beam. In solid and thin wall closed sections these stresses can often be neglected. However, in thin wall open sections these axial stresses can be large. Moreover, the stiffness of thin wall open sections is strongly increased due to warping restraints.

Differential Equation

In 1940, V.Z. Vlasov developed a torsion theory in which restrained warping is included [9]. This theory is also called “warping torsion” or “non-uniform torsion”. Next to this the torsion theory of De Saint Venant is also called “circulatory torsion” or “uniform torsion”. In the theory of Vlasov the specific torsion θ is not constant along the *x*-axis. The rotation φ of the beam cross-section follows from the differential equation

$$EC_w \frac{d^4\varphi}{dx^4} - GI_t \frac{d^2\varphi}{dx^2} = m_x \, ,$$

where GI_t is the torsion stiffness, EC_w is the warping stiffness and m_x is a distributed torsion moment along the beam. The warping constant C_w has the unit m^6 and is defined as

$$C_w = \int_A \psi^2 dA \, .$$

The bi-moment is defined as

$$B = -\int_A \sigma_{xx} \psi dA \, .$$

It occurs in a cross-section when warping is restrained. It has the unusual unit Nm².

For I-sections the bi-moment B can be interpreted as the moment M in each of the flanges times their distance a (Fig. 4). For other sections the interpretation is not this simple. In general the bi-moment is the distribution of axial stresses that is needed to reduce the warping of the section.

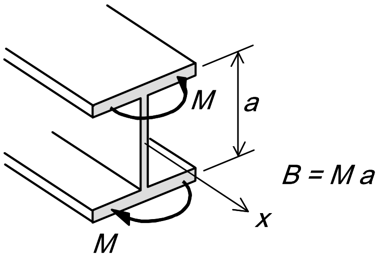


Figure 4. Bi-moment in an I-section

At the boundary either the rotation φ is imposed or the torsion moment loading M_t is imposed. At the same time the warping $\dfrac{d\varphi}{dx}$ or the bi-moment B is imposed.

Examples of boundary conditions are.

Fixed support, No rotation and no warping $\varphi = 0, \quad \dfrac{d\varphi}{dx} = 0$

Fork support, No rotation and free warping $\varphi = 0, \quad B = 0$

Free end, Free rotation and free warping $M_t = 0, \quad B = 0$

$$B = -EC_w \frac{d^2\varphi}{dx^2}$$

$$M_t = GI_t \frac{d\varphi}{dx} + \frac{dB}{dx}$$

The Vlasov theory reduces to the theory of De Saint Venant if the warping stiffness is zero, the distributed moment is zero and warping is free.

Example of a Box-girder bridge

A box-girder bridge has a length l = 60 m. The cross-section dimensions and properties are shown in Figure 5.[1] The concrete Young's modulus is $E = 0.30\ 10^{11}$ N/m² and Poisson's ratio $\nu = 0.15$. At both ends the bridge is supported while warping is free. In the middle the bridge is loaded by a torque $T = 269\ 10^5$ Nm. This loading occurs when the bridge is supported at mid span by two temporary columns of which one fails due to an accident. Therefore, the torque T is due to the support reaction of the remaining temporary column.

[1] This example is adapted from lecture notes by dr.ir. C. van der Veen on reinforced and prestressed concrete design for the Dutch Concrete Association.

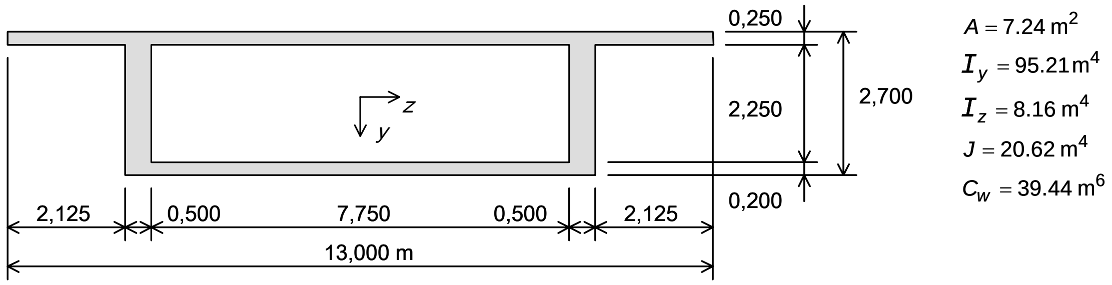
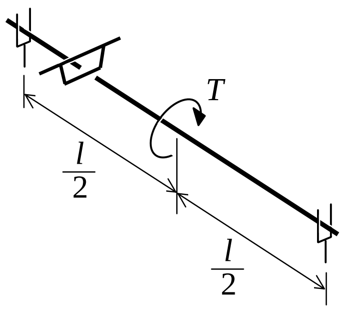


Figure 5. Bridge cross-section

Due to symmetry we consider half the bridge.



The boundary conditions at the support $x = 0$ are

$$\varphi = 0, \quad \frac{d^2\varphi}{dx^2} = 0 .$$

The boundary conditions in the middle $x = \frac{1}{2}l$ are

$$GI_t \frac{d\varphi}{dx} - EC_w \frac{d^3\varphi}{dx^3} = \frac{1}{2}T, \quad \frac{d\varphi}{dx} = 0 .$$

The differential equation is solved by Maple (Appendix 1) (Figure 6, 7 en 8).

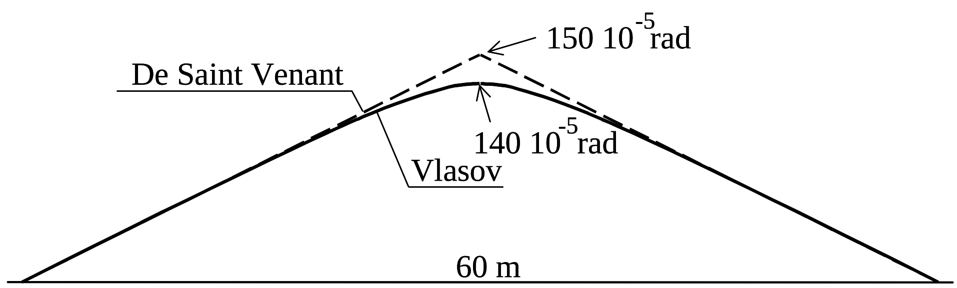


Figure 6. Rotation φ

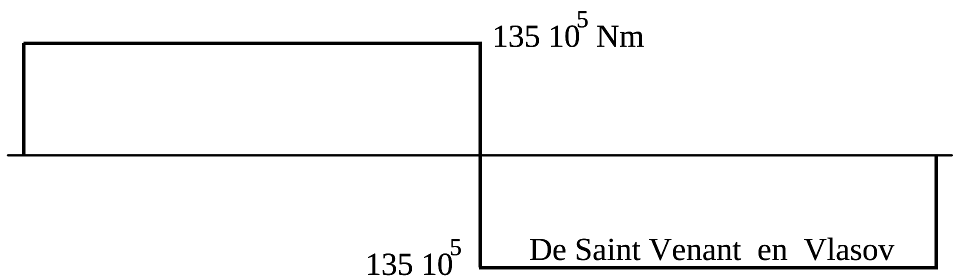


Figure 7. Torsion moment distribution M_t

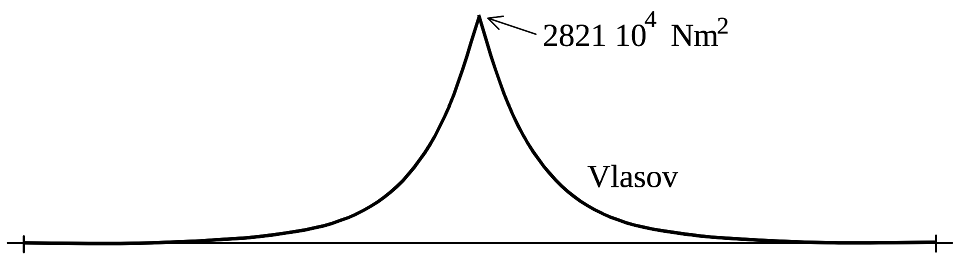


Figure 8. Bi-moment distribution *B*

Vlasov Stresses

The stress distribution in the Vlasov theory consists of three parts. 1) the shear stress according to the theory of De Saint Venant, 2) shear stress due to restrained warping en 3) axial stresses due to restrained warping. In general the largest values of the parts occur at different locations in a cross-section. Therefore, software is needed for locating the decisive stress. This is even more so if also stresses occur due to 4) axial force N , 5) moment M_y in the *y*-direction, 6) moment M_z in the *z*-direction, 7) shear V_y in the *y*-direction and 8) shear V_z in the *z*-direction.

If *y* and *z* are the principal directions of the cross-section than the axial stresses are calculated by

$$\sigma_{xx} = \frac{N}{A} + \frac{M_y}{I_y} z - \frac{M_z}{I_z} y + \frac{B}{C_w} \psi$$

For the shear stresses in thin wall cross-sections also formulas exist. However, these are to elaborate to include in this text. As far as the author knows no formulas exist for the shear stresses σ_{xy} and σ_{xz} due to restrained warping in solid cross-sections exist.

Example on Stresses in a Box-girder Bridge

We consider the box-girder bridge of the previous example. Using the program ShapeBuilder we can compute the warping function ψ and the torsion properties I_t and C_w of the cross-section (Fig. 9). The largest value of ψ is 51182 cm² in the right-hand lower corner of the box-grider. Previously the largest bi-moment $B = 2821\ 10^4\ \text{Nm}^2$ was calculated (Fig. 8). The associated largest axial stress due to warping is

$$\sigma_{xx} = \frac{B}{C_w} \psi = \frac{2821\ 10^4\,\text{Nm}^2}{39.44\ \text{m}^6} 5.118\ \text{m}^2 = 3.66\ \ 10^6\ \text{N/m}^2 = 3.66\ \text{N/mm}^2$$

This is small compared to stresses due to bending. Generally, in solid and closed sections the stresses due to restrained warping can often be neglected. For thin wall open sections these stresses often cannot be neglected.

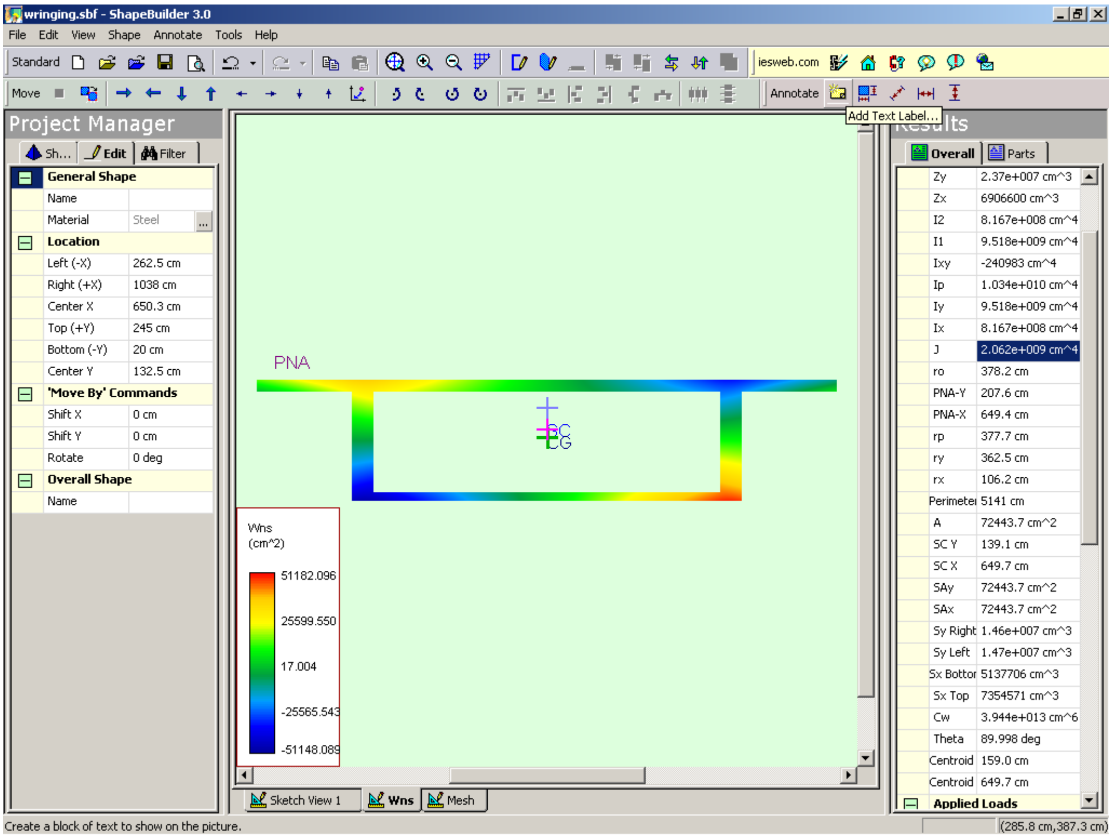
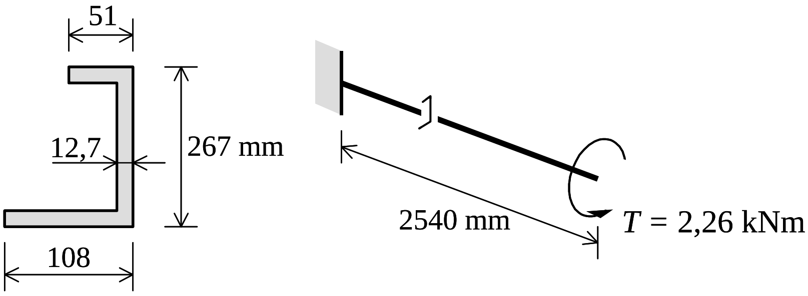


Figure 9. Warping function ψ of the bridge

Example of Stresses in a Thin Wall Beam

We consider the cross-section of Figure 23. The beam is fixed at one end at which warping is prevented. At the other end the beam is loaded by a torque T while warping can occur freely [11]. The material and cross-section data of the beam are $E = 207000 \text{ N/mm}^2$, $G = 79300 \text{ N/mm}^2$, $I_t = 278000 \text{ mm}^4$ and $C_w = 191\ 10^8 \text{ mm}^6$.



Figuur 10. Thin wall beam

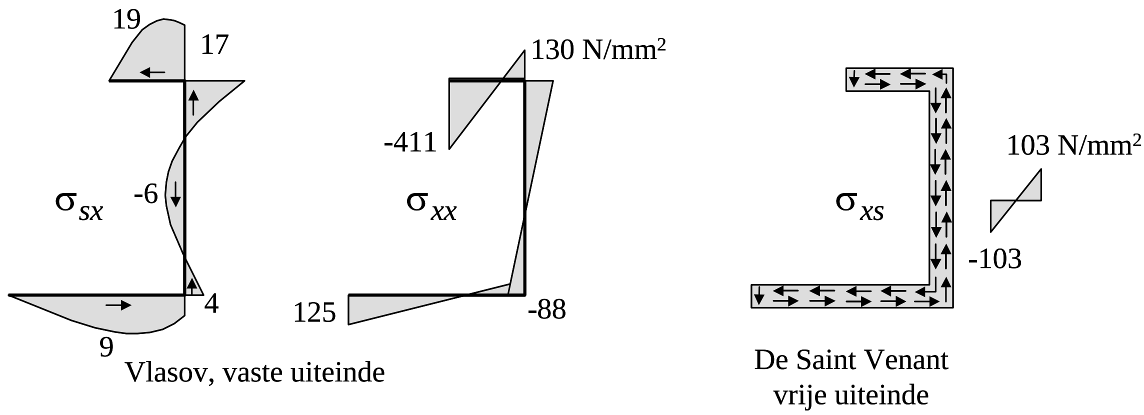


Figure 11. Stresses in the thin wall beam [11]

Figure 11 shows the Vlasov stresses that occur at the beam support. The Saint Venant stresses occur at the loaded end of the beam. Note that the largest Vlasov axial stress σ_{xx} is much larger than the largest Saint Venant shear stress σ_{xs} (in absolute sense). The Vlasov shear stresses σ_{xs} are small, nonetheless the moment which they produce is equal to the loading T.

Software

In frame programs that use the Vlasov theory not only the torsion stiffness GI_t but also the warping stiffness EC_w of every member needs to be known. Table 10 gives the torsion properties of some thin wall cross-sections. In addition we need to enter whether the warping is prevented, coupled or free in each member end.

In the appendix the stiffness matrix is included for implementing the Vlasov theory in a frame analysis program. A degree of freedom – the warping – and a loading – the bi-moment – need to be added to the program for every element end.

O = shear center $\quad J$ = torsion constant $\quad C_w$ = warping constant

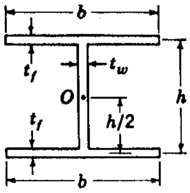
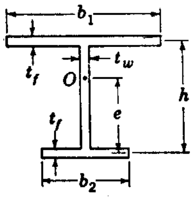
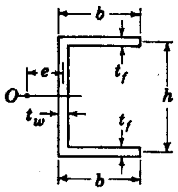
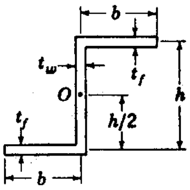
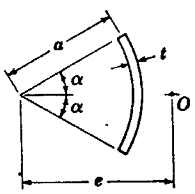
	$J = \dfrac{2bt_f^3 + ht_w^3}{3}$ $C_w = \dfrac{t_f h^2 b^3}{24}$	If $t_f = t_w = t$: $J = \dfrac{t^3}{3}(2b + h)$
	$e = h\dfrac{b_1^3}{b_1^3 + b_2^3}$ $J = \dfrac{(b_1 + b_2)t_f^3 + ht_w^3}{3}$ $C_w = \dfrac{t_f h^2}{12}\dfrac{b_1^3 b_2^3}{b_1^3 + b_2^3}$	If $t_f = t_w = t$: $J = \dfrac{t^3}{3}(b_1 + b_2 + h)$
	$e = \dfrac{3b^2 t_f}{6bt_f + ht_w}$ $J = \dfrac{2bt_f^3 + ht_w^3}{3}$ $C_w = \dfrac{t_f b^3 h^2}{12}\dfrac{3bt_f + 2ht_w}{6bt_f + ht_w}$	If $t_f = t_w = t$: $e = \dfrac{3b^2}{6b + h}$ $J = \dfrac{t^3}{3}(2b + h)$ $C_w = \dfrac{tb^3h^2}{12}\dfrac{3b + 2h}{6b + h}$
	$J = \dfrac{2bt_f^3 + ht_w^3}{3}$ $C_w = \dfrac{b^3h^2}{12(2b + h)^2} \times [2t_f(b^2 + bh + h^2) + 3t_w bh]$	If $t_f = t_w = t$: $J = \dfrac{t^3}{3}(2b + h)$ $C_w = \dfrac{tb^3h^2}{12}\dfrac{b + 2h}{2b + h}$
	$e = 2a\dfrac{\sin\alpha - \alpha\cos\alpha}{\alpha - \sin\alpha\cos\alpha}$ $J = \dfrac{2a\alpha t^3}{3}$ $C_w = \dfrac{2ta^5}{3} \times \left[\alpha^3 - \dfrac{6(\sin\alpha - \alpha\cos\alpha)^2}{\alpha - \sin\alpha\cos\alpha}\right]$	If $2\alpha = \pi$: $e = \dfrac{4a}{\pi} \qquad J = \dfrac{\pi a t^3}{3}$ $C_w = \dfrac{2ta^5}{3}\left(\dfrac{\pi^3}{8} - \dfrac{12}{\pi}\right) = 0.0374ta^5$

Table 10. Torsion properties of thin wall cross-sections [10]. Note that J is the torsion constant I_t .

Trick

By far most frame analysis programs apply the torsion theory of De Saint Venant There is a trick to include the effect of restrained warping in these programs. When both member ends cannot warp the torsion stiffness needs to be multiplied by an enlargement factor

$$\frac{\beta}{\beta - 2} \qquad \beta > 5 .$$

When one of the member ends cannot warp the torsion stiffness needs to be multiplied by the factor

$$\frac{\beta}{\beta - 1} \qquad \beta > 3 .$$

These formulas are 1% accurate. For small values of β also exact formulas exist in reference [12]. In these formulas the distributed torsion moment loading is not present $m_x = 0$. For coupled warping no trick exists.

Safe or Unsafe?

Almost all frame programs in practice use De Saint Venant torsion theory ignoring the effects of restraint warping. In this chapter it was shown that real structures are stiffer than the torsion theory of De Saint Venant assumes. Therefore, the real deformations will be smaller than the computed ones. Consequently, for the serviceability limit state the traditional frame analysis is safe.

Also it was shown that locally the stresses can be much larger than the Saint Venant predicts. However, this does not mean that the involved structural member will collapse. After all, many structural materials are somewhat plastic (steel, timber, reinforced concrete). According to plasticity theory every equilibrium system that does not violate the yield strength is a safe approximation of the carrying capacity of the structure. A linear elastic computation according to the theory of De Saint Venant is such an equilibrium system. Consequently, also for the ultimate limit state the traditional frame analysis is safe.

Literature

9. V.Z. Vlasov, “Thin-Walled Elastic Bars” (in Russian), 2nd ed., Fizmatgiz, Moscow, 1959.
10. S.P. Timoshenko, J.M. Gere, “Theory of Elastic Stability”, McGraw-Hill, New York, second edition 1961.
11. W.F. Chen, T. Atsuta, “Theory of Beam-Columns, Vol. 2: Space Behaviour and Design”, McGraw-Hill, New York, 1977.
12. P.C.J. Hoogenboom, A. Borgart, "Method for including restrained warping in traditional frame analyses", HERON, Vol. 50 (2005) No. 1, pp. 55-68.

Appendix 1

Maple calculation of the differential equation of the box-girder bridge

```
> restart:
> l:=60:          # [m]
> ECw:=1183e9:    # [Nm4]
> GIw:=2690e8:    # [Nm2]
> mx:=0:          # [Nm/m]
> T:=269e5:       # [Nm]
>
> with(DEtools):
> ODE:=ECw*diff(phi(x),x,x,x,x)-GIw*diff(phi(x),x,x)=mx;
```

$$ODE := .1183\ 10^{13} \left(\frac{\partial^4}{\partial x^4} \phi(x) \right) - .2690\ 10^{12} \left(\frac{\partial^2}{\partial x^2} \phi(x) \right) = 0$$

```
> bound_con:= phi(0)=0, (D@@2)(phi)(0)=0, GIw*D(phi)(l/2)-
ECw*(D@@3)(phi)(l/2)=T/2, D(phi)(l/2)=0;
```

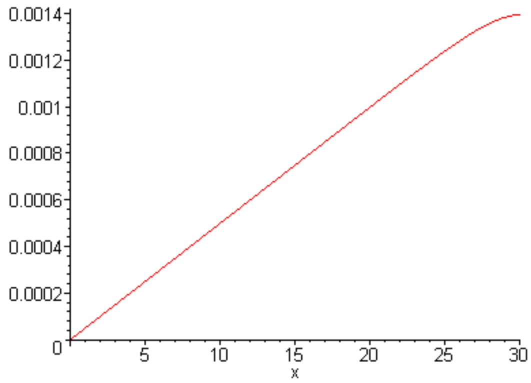
$$bound\_con := \phi(0) = 0, (D^{(2)})(\phi)(0) = 0,$$

$$.2690\ 10^{12}\ D(\phi)(30) - .1183\ 10^{13}\ (D^{(3)})(\phi)(30) = .1345000000\ 10^{8}, D(\phi)(30) = 0$$

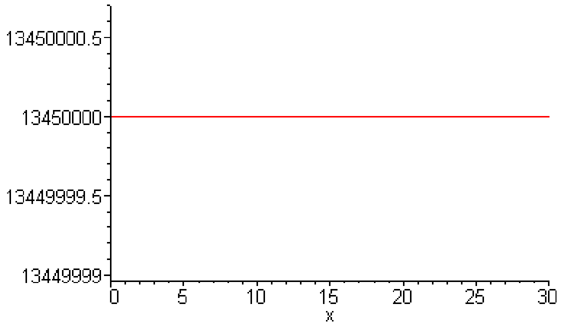
```
> evalf(dsolve({ODE,bound_con},{phi(x)}));
```

$$\phi(x) = .00005000000000\ x + .6423299796\ 10^{-10}\ e^{(-.4768521749\ x)} - .6423299796\ 10^{-10}\ e^{(.4768521749\ x)}$$

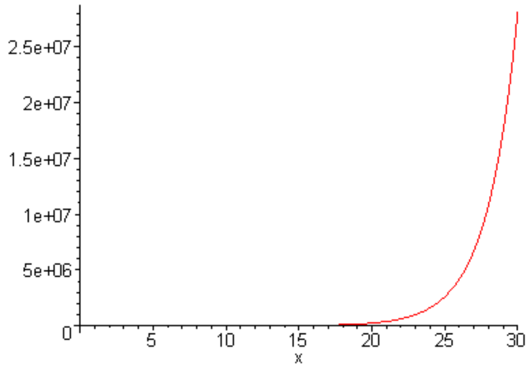
```
> phi:=0.5000000000e-4*x-0.6423299796e-
10*exp(0.4768521749*x)+0.6423299796e-10*exp(-0.4768521749*x):
> Mo:=-ECw*diff(phi,x,x):
> Mw:=Re(GIw*diff(phi,x)+diff(Mo,x)):
> plot(phi(x),x=0..l/2);
```




```
> plot(Mw,x=0..l/2);
```




```
> plot(Mo,x=0..l/2);
```




```
> x:=l/2: simplify(Mo); simplify(phi);
```

$$.2820580643\ 10^{8}$$

$$.001395145701$$

Appendix 2

Stiffness Matrix for the Vlasov Torsion Theory

$$x = 0 \quad \rightarrow \quad \varphi_1 = \varphi\,, \quad \theta_1 = \frac{d\varphi}{dx}, \quad T_1 = -M_t\,, \quad B_1 = B$$

$$x = l \quad \rightarrow \quad \varphi_2 = \varphi\,, \quad \theta_2 = \frac{d\varphi}{dx}, \quad T_2 = M_t\,, \quad B_2 = -B$$

$$\begin{bmatrix} T_1 \\ T_2 \\ B_1 \\ B_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} l \\ \frac{1}{2} l \\ \frac{1}{2}\frac{l^2}{\beta^2\lambda} \\ -\frac{1}{2}\frac{l^2}{\beta^2\lambda} \end{bmatrix} m_x = \begin{bmatrix} \frac{GI_t}{l}\eta & -\frac{GI_t}{l}\eta & GI_t\lambda & GI_t\lambda \\ -\frac{GI_t}{l}\eta & \frac{GI_t}{l}\eta & -GI_t\lambda & -GI_t\lambda \\ GI_t\lambda & -GI_t\lambda & \frac{EC_w}{l}\xi & \frac{EC_w}{l}\mu \\ GI_t\lambda & -GI_t\lambda & \frac{EC_w}{l}\mu & \frac{EC_w}{l}\xi \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \theta_1 \\ \theta_2 \end{bmatrix}$$

$$\eta = \frac{\beta(\mathrm{e}^{\beta}+1)}{\beta + 2 + (\beta - 2)e^{\beta}}$$

$$\lambda = \frac{\mathrm{e}^{\beta}-1}{\beta + 2 + (\beta - 2)e^{\beta}} \qquad\qquad \beta = l\sqrt{\frac{GI_t}{EC_w}}$$

$$\mu = \frac{\beta}{\mathrm{e}^{\beta}-1}\;\frac{e^{2\beta} - 2\beta e^{\beta} - 1}{\beta + 2 + (\beta - 2)e^{\beta}}$$

$$\xi = \frac{\beta}{\mathrm{e}^{\beta}-1}\;\frac{\beta + 1 + (\beta - 1)e^{2\beta}}{\beta + 2 + (\beta - 2)e^{\beta}}$$